)f these the last three are variable, the others fixed. The following binary ɪfixes are connectors (i.e. they form sentences from other sentences)

$$\Rightarrow, \rightarrow, \rightleftarrows, \& .$$

To avoid excessive parentheses we agree that all binary operators ɪnd connectors are associative to the left. Occasionally we shall use the .ot notation. In that case we shall follow the rules of [UDB] as modified ɪ § **2**B5.

Various affixes may be attached to these functional symbols. The above onventions apply regardless of these affixes.

When superscripts and subscripts are attached to the same base symbol he superscripts are to be taken as senior. Thus $\mathsf{B}_m{}^n$ is to be read as $(\mathsf{B}_m)^n$.

Quotation marks

We shall follow established practice in using a specimen of a symbol ɪr expression, enclosed in single quotes, as a name for that symbol (or xpression). This is rather a technical use of quotation marks; we thereore reserve single quotes for that purpose and employ ordinary (double) ɪuotes for the nontechnical uses. (Cf. § **1**D1.)

Additions during printing

Additions made to the text while in proof are either explicitly so mar:ed or are enclosed in square brackets.

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Introduction

Combinatory logic is a branch of mathematical logic which concerns itself with the ultimate foundations. Its purpose is the analysis of certain notions of such basic character that they are ordinarily taken for granted. These include the processes of substitution, usually indicated by the use of variables; and also the classification of the entities constructed by these processes into types or categories, which in many systems has to be done intuitively before the theory can be applied. It has been observed that these notions, although generally presupposed, are not simple; they constitute a prelogic, so to speak, whose analysis is by no means trivial.

Two questions have incited the making of this analysis. The first of these is the problem of formulating the foundations of logic as precisely as possible; and in a manner which is simple, not only from the standpoint of structure, but also from that of meaning. This is an end worthy of being pursued for its own sake, and one to which some of the greatest minds have devoted a good share of their energies. The exhibition of the complicated rules of the prelogic as a synthesis made from rules of much simpler character is a step in that direction. The second question is the explanation of the paradoxes. There is reason for believing that the true source of our difficulties with these puzzles lies in the prelogic itself, and therefore that an analysis of the prelogic will contribute to understanding them.

In order to get a better idea of the motivation and purpose of combinatory logic, it will be well to elaborate these points a little before we proceed.[1]

A. THE ANALYSIS OF SUBSTITUTION

Consider the formulation of the propositional algebra in the Principia Mathematica.[2] Let us use the symbols 'p', 'q', 'r', etc. for unspecified

1. For general summaries of combinatory logic from the standpoint of the present work see [CFM], also [TCm] and [LCA]; a more extensive summary is given in Cogan [FTS]. Expositions of certain parts may be found in Feys [TLC], Rosenbloom [EML, pp. 109–133], and Rosser [DEL]. For the underlying philosophy and motivation (supplementing somewhat the discussion of this Introduction) reference may also be made to Schönfinkel [BSM] and the introductory portions of Rosenbloom (l.c., pp. 109–111); see also [PBP], [GKL] pp. 509–518, [ALS] pp. 363–374, [AVS] pp. 381–386, and [FPF] pp. 371–375. For a possible application see [LPC].
2. [PM.I] Part I. Sec. A.

variables of the system, 'P', 'Q', 'R', for unspecified constructs from the variables by the operations. The system, when certain defects in the formalization have been remedied,[3] has two rules: (1) a "rule of inference" (modus ponens), which we may write

(1) $$\frac{\vdash P \qquad \vdash \neg P \vee Q}{\vdash Q,}$$

and (2) a rule of substitution, viz., that

(2) $$\frac{\vdash Q}{\vdash R}$$

is an allowable inference whenever R is obtained from Q by substitution of certain constructs $P_1, P_2, \ldots, P_m$ for the variables $p_1, p_2, \ldots, p_m$ appearing in Q. It is evident that the second rule here is vastly more complicated than is the first. We shall examine the nature of this complexity more closely.

Let us look on the system, not as a formalism, but as a series of statements with a meaning. What, then, is the meaning of a statement

(3) $$\vdash P,$$

for example

$$\vdash \neg p \vee (p \vee q)\ ?$$

Certainly the statement concerns neither the symbols 'p', 'q' nor any objects which they denote. Rather, what is asserted is a relation between the operations of negation and alternation. Thus the P in an assertion (3) is a combination of the primitive operations concerning which we assert a certain property (viz., that it is a tautology). Such combinations P we shall call, for the moment, "functions".

Let us examine the rules (1) and (2) from this standpoint. The connection between the function Q and the functions P and $\neg P \vee Q$ in (1) is quite precise and definite. Not so the connection between Q and R in (2). If we admit only modes of combination which are as simple and definite as that in (1), there are really infinitely many such modes implicit in the rule (2) even for the case $m = 1$.[4] For it must be understood as part of

3. The principal defect was the omission of the rule of substitution, and in general the neglect of the distinction between a formal axiom and a rule. This defect has been commented on by several persons, including Russell himself [IMP, p. 151]. For improved presentations see, for example, [HA], Post [IGT], Feys [LGL], Herbrand [RTD]. Here we shall follow [HA]; but we shall conform to the Principia notation except for the use of '$\neg$' instead of the Principia '$\sim$' (the former symbol, introduced by Heyting [FRI], is preferred because it is more specific).

4. It is known the general case can be reduced to the case $m = 1$. See § **2C2**.

the concept of a function that there is a fixed order of its variables; thus $p \vee q$ as a function of p and q is a distinct function from $q \vee p$. The substitution of R for the ith variable in Q is, as mode of combination of functions, distinct from its substitution for the kth variable if $i \neq k$; and in each of these cases there is a further multiplicity of distinct processes according to the number of arguments in Q. There are also various possibilities for identification and permutation of the variables to be taken into account. Moreover, when these processes of construction are iterated, there are equivalences which have to be taken intuitively.

These complexities may be illustrated in the field of ordinary mathematics as follows. Let d and s be the difference and square functions, and let the letters 'x', 'y', 'z' represent the first, second, third arguments respectively. Then d is $x - y$ and s is x^2. The substitution of s for the first argument of d is $x^2 - y$, for the second argument is $x - y^2$. If d is substituted for one of its own arguments there is an ambiguity as to how the new arguments are to be numbered; let us suppose the arguments of the substituted function are kept consecutive and interpolated as a group in the sequence of arguments of the base function. Then by substitution of d in the first and second arguments of d we have respectively the functions $(x - y) - z$ and $x - (y - z)$. By permutation and identification of arguments we have further functions such as $x - (x - y)$, $y - (x - z)$, etc. Finally, if we substitute s for the first argument in d and then again for the second argument in the result, we get the same function, viz. $x^2 - y^2$, as if we had performed the substitutions in reverse order.[5]

If we pass from this intuitive point of view to a more formal one, then it becomes a problem how to define the substitution operation exactly. A rigorous answer to this question requires, as we shall see, a definition by recursion. The conclusion of the previous paragraph still stands, viz., that we have in rule (2) an immensely more complicated rule than in (1).[6] Moreover, the recursive definition does not have any bearing on the question of simplicity in relation to meaning.

All this holds for what is generally considered the simplest of logical systems, the propositional algebra. If we pass to cases where there is more then one category of variables, and where some of the variables may be bound, the situation becomes more complicated still. The extent of the complications in such cases may be seen from the fact that most formulations of the rule for substitution for a functional variable in the first-order predicate calculus which were published, even by the ablest logicians, before 1940, were demonstrably incorrect; and there is little doubt that one of the first correct formulations, that given by Church

5. Cf. [ALS], pp. 368–9.

6. Cf. the formulation in Rosenbloom [EML, pp. 40 ff, 182 ff].

[IML], p. 57, was derived by the aid of the theory of lambda-conversion, the form of combinatory logic which is his specialty.[7]

We shall see that in combinatory logic all these processes can be formulated on the basis of a logical system of great simplicity. This system is of finite structure, in a very strong sense, and its rules are of the same order of complexity as modus ponens. This represents, therefore, an advance toward the first objective.

B. THE RUSSELL PARADOX

So much for the first point. We turn now to the second.

Consider, for example, the paradox of Russell. This may be formulated as follows: Let $F(f)$ be the property of properties f defined by the equation

$$F(f) = \neg f(f), \tag{1}$$

where '$\neg$' is the symbol for negation. Then, on substituting F for f, we have

$$F(F) = \neg F(F) . \tag{2}$$

If, now, we say that $F(F)$ is a proposition, where a proposition is something which is either true or false, then we have a contradiction at once. But it is an essential step in this argument that $F(F)$ should be a proposition. This is a question of the prelogic; in most systems it has to be decided by an extraneous argument.

The usual explanations of this paradox are to the effect that F, or at any rate $F(F)$, is "meaningless". Thus, in the Principia Mathematica the formation of $f(f)$ is excluded by the theory of types; in the explanation of Behmann [WLM] one cannot use (1) as a definition of F because the "Kurzzeichen" 'F' cannot be eliminated. By such methods one can, presumably, exclude the paradoxes from a given system. But there is evidently something about the preceding intuitive argument which is not explained by such exclusions.

In combinatory logic we must make, in order to achieve the objectives already mentioned, the following demands: (a) there shall be no distinction between different categories of entities,[8] hence any construct formed

7. At least one can demonstrate the correctness of Church's formulation by that method. One should note, however, the historical statement made by Church l.c. p. 63, and the correction thereto in his [rev HA]. In the latter Church admits that he was mistaken in attributing an error to the formulation [HB.I], and that the latter is probably the first published correct statement. On account of this admission the statements in Rosenbloom [EML] p. 109, and Curry [rev Z] (which is based on the claim made in Church [IML])are inaccurate.

8. If the system has variables, a distinction between variables and other entities may be allowed; but there shall be only one kind of variable, and any entity can be substituted for any variable. Such systems have an intermediate character. (See § C.)

from the primitive entities by means of the allowed operations must be significant in the sense that it is admissible as an entity;[9] (b) there shall be an operation corresponding to application of a function to an argument; (c) there shall be an equality with the usual properties; and (d) the system shall be *combinatorially complete*,[10] i.e., such that any function we can define intuitively by means of a variable can be represented formally as an entity of the system.

From these four demands it follows, not only that the F defined by (1) is significant, but also that the equation (2) is intuitively true. This is by no means an objection to the system; on the contrary it is a step in advance. We can no longer "explain" a paradox by running away from it; we must stand and look it in the eye. Something is gained by the mere bringing about of this state of affairs. The paradoxes are forced, so to speak, into the open, where we can subject them to analysis. This analysis must explain the fact that $F(F)$ does not belong to the category of propositions, an explanation which comes within the province of combinatory logic as here conceived.

C. PLAN OF THE WORK

The subject matter sketched in the foregoing discussion falls naturally into two main parts. The first part is the analysis of the substitution processes in themselves, without regard to the classification of entities into categories. The second part introduces the machinery for effecting a classification into categories, and also relations to special logical notions such as implication, universal quantification, etc. Naturally the first part has a more intimate relation to the first of the above objectives, whereas the second part is more directly concerned with the second objective; but there is some overlapping.

In the analysis a basic role is played by certain operators which represent combinations as functions of the variables they contain (perhaps along with other variables). The combinations in question are those formed from the variables alone by means of the operation postulated in the second of the above demands. By the requirement of combinatorial completeness these operators are represented by certain entities of the system. These entities, and combinations formed from them by the

9. I.e., the system must be completely formalized in the sense of § 1E (except for the circumstance in the preceding footnote).

10. Rosenbloom (l.c., p. 116) uses the term 'functionally complete'. This is not, however, the ordinary sense of that term. Thus it is usually said that the classical propositional algebra is functionally complete, although no functions whatever can be represented in it in the sense we are here discussing. The term 'combinatorially complete' was introduced in [PKR], p. 455.

postulated operation, are called *combinators*. The term '*combinatory logic*' is intended to designate that part of mathematical logic which has essential reference to combinators,[11] including all that is necessary for an adequate foundation of the more usual logical theories.

It will be seen that the first part of combinatory logic deals with the combinators in their formal relations to one another with reference only to a relation of equality; this part can therefore be called *pure combinatory logic*. The second part deals with combinators as included in some specified categories, as affected by quantifiers, and so on; we shall call it *illative combinatory logic*.

The combinators themselves may be defined in terms of an operation of abstraction, or certain of them may be postulated as primitive ideas and the others defined in terms of them. The first alternative leads to the *calculus of lambda-conversion* of A. Church, and various modifications of it; the second leads to the (*synthetic*) *theory of combinators*.[12] It is the synthetic theory which gives the ultimate analysis of substitution in terms of a system of extreme simplicity. The theory of lambda-conversion is intermediate in character between the synthetic theories and ordinary logics. Although its analysis is in some ways less profound—many of the complexities in regard to variables are still unanalyzed there—yet it is none the less significant; and it has the advantage of departing less radically from our intuitions. Accordingly we have decided, without regard to historical considerations, to treat the various forms of lambda-calculus first, and to develop the synthetic theories afterwards.

We are thus following the order of an analysis; the most fundamental formulation is obtained at the end of Chapter **7**. The theory of the λ-calculuses will be found in Chapters **3–4**, while Chapter **5** forms a transition between the λ-calculuses and the synthetic theories of Chapters **6–7**. The equivalence between a synthetic theory and the corresponding λ-calculus is established in Chapter **6**; so that developments from that point on can be based on either foundation. The consistency theorem of Church and Rosser is considered in Chapter **4**; certain generalizations of this theorem will be considered at the same time.

An important aspect of pure combinatory logic, called *combinatory arithmetic*, is concerned with the relations between combinators and various kinds of numbers. This is a development which was not anticipated at the beginning. The original suggestions were made by Church,

11. The choice of the term 'combinatory', in preference to 'combinatorial', is thus in agreement with the Oxford English Dictionary.

12. The term 'theory of combinators' might well replace the term 'pure combinatory logic'. However the uses of the former term in the past have been as in the text, and it would probably be confusing to change it. The word 'synthetic' will be added when it is desired to be explicit.

[SPF. II, p. 863] and some contributions were made by Rosser [MLV]; but the main developments are due to Kleene.[13] It has turned out, not only that one can make acceptable definitions of the positive (or natural) integers in terms of combinators, but that every function of natural integers which is "general recursive" (in the sense of a theory developed by Kleene from suggestions of Herbrand and Gödel) is definable by means of combinators.[14] This criterion has been shown to be equivalent to a computability notion introduced by Turing [CNA]. The mutual equivalence of these three apparently so different notions is ground for believing that we have in them a definition of an effectively calculable function.[15] If so, then every effectively calculable numerical function can be defined in terms of combinators and conversely. Some noteworthy developments on the basis of combinatory arithmetic were obtained by Church in the 1930's.[16]

Our original intention was to issue the present work in two volumes, of which the first dealt with pure combinatory logic, and the second with illative. For reasons of expediency, which are none the less good and sufficient, we have had to change these plans. In particular we have postponed our treatment of combinatory arithmetic to the second volume.[17] In its place we have included three chapters on illative combinatory logic. The first of these chapters, Chapter **8**, is an introduction to illative combinatory logic as a whole; the others, Chapters **9** and **10**, deal with the first phase of illative combinatory logic, which is called the theory of functionality. For further introductory discussion, the reader is referred to Chapter **8**.

13. See his [TPI], [LDR], and to some extent also his [PCF]. Unfortunately these papers were written in the notation of Church [SPF]; this notation was cumbersome, and Church himself abandoned it about **1940** (cf. § **3S2**). For accounts of Kleene's work, using the Schönfinkel type of notation, see Church [CLC] and Rosser [DEL].

14. Here by "function" is meant a single valued function which is defined for every natural integer and has a natural integer as its value. The converse of the statement also holds for any such function. There is also a similar statement for "partial" functions; but we are not attempting to be precise about it here.

15. For recursiveness and its relation to the work of Turing, Post, and others see Kleene [IMM]. This contains ample references to the original papers. It can be supplemented in some respects by Péter [RFn]. There is also a forthcoming book by Martin Davis, and one by J. Myhill and J. C. E. Dekker. For the relation of recursiveness to effective calculability see especially Kleene l c. § 62.

16. See his [RPx], [PFC], [UPE], [MLg], [NEP], [CNE]. Most of these papers were also written in the notation of his [SPF], and the most extensive of them, [MLg], is not easily accessible. The account of these matters in his [CLC] contains some improvements in detail, but is disappointingly brief. There is also a theory of constructive ordinal numbers in terms of combinators; for this see Church and Kleene [FDT], Church [CSN], and Kleene [FPT], [NON].

17. On this account the discussion of combinatory arithmetic in the preceding paragraph is fuller than might otherwise be expected. As a matter of fact, our present research activity is along lines so near the foundations that combinatory arithmetic has not yet begun to play a role in it. We have at present writing relatively little to add to what is contained in the preceding references beyond what appears in [PKR] and [FRA].

Before entering upon combinatory logic proper in Chapter **3**, it will be necessary to formulate in a precise way certain notions connected with the methodology of formal reasoning. The concepts of logical system, entity of such a system, rule, variable, etc., which have entered into the foregoing discussion, are necessarily somewhat vague; we shall have to make them more precise in order to have a sound basis on which to proceed. This will concern us in Chapters **1** and **2**.

D. HISTORICAL SKETCH

Combinatory logic began with Schönfinkel [BSM] in 1924.[18] In that paper, which was prepared for publication by Behmann, Schönfinkel called attention to the desirability of eliminating variables from logic, introduced the idea of application, and showed that functions of different numbers of variables could be eliminated by means of it—provided the idea of function were enlarged so that functions could be arguments as well as values of other functions (cf. § **3**A). He introduced the special combinators later called **I**, **K**, **C**, **B**, **S** (he called them *I*, *C*, *T*, *Z*, *S*, respectively), and showed that logical formulas could be expressed without variables by means of **S** and **K**. But Schönfinkel gave no technique of deduction, and, in particular, gave no means of proving formally that two intuitively equivalent combinators are equal.[19]

A deductive theory somewhat along the lines indicated by Schönfinkel was obtained in [GKL], for which [ALS] was a preliminary. This was a synthetic theory based on a set of primitive combinators differing from those of Schönfinkel. In this paper the terms 'combinator' and 'combinatory logic' were introduced. The proofs were quite long and clumsy compared to those found later. But the consistency of the formulation and its sufficiency for pure combinatory logic were established.

In the meantime, and quite independently, Church was developing the system of formal logic which finally appeared in his two papers [SPF.I] and [SPF.II]. In this theory a certain λ-operation, representing the abstraction of a function from its unspecified value, so to speak, played a central role. The result is that Church's theory contained combinators

18. For some earlier premonitions see the next to last paragraph of this section.
19. Schönfinkel also had some ideas in regard to illative combinatory logic. He introduced an "Unverträglichkeitsfunktion" *U* which has the same relation to the Sheffer stroke function that formal implication does to ordinary implication. He showed that the ordinary stroke function and quantifiers could be defined in terms of *U*. (This anticipates, in a way, the use of formal implication as a primitive in Quine [SLg]). Schönfinkel then maintained that every logical statement could be expressed in terms of **S**, **K**, and *U* without postulating the notions of proposition or propositional function. But more recent developments in illative combinatory logic show that any reasonable system constructed on so naive a basis is inconsistent.

(according to the definition above), and therefore belonged to combinatory logic. The system also contained implication, etc.; consequently it belonged to illative combinatory logic. However, the part of his theory which concerned pure combinatory logic soon crystallized out, and developed into his calculus of lambda-conversion. Important theorems of undecidability were derived in this system by Church and his co-workers; and the system was put into relation with recursive number theory, the theory of ordinal numbers, the predicate calculus, etc.[20] Some of these theorems will concern us in the later chapters of this work.

It was soon evident that there was some relation between the theories of Curry and Church. The exact relationship was investigated by Rosser in his thesis [MLV]. He constructed a synthetic theory of combinators, weaker in some respects than Curry's, which was equivalent to Church's λ-conversion.[21] Conversely it was shown later that a strengthened theory of λ-conversion was equivalent to Curry's synthetic theory. From that time on it was clear that a theory of combinators could be expressed as a theory of λ-conversion and vice-versa. Some important improvements in the synthetic theories are due essentially to suggestions made by Rosser, both in his thesis and later.[22]

An event which had an important influence on the further progress of the subject was the discovery, by Rosser and Kleene working con-

20. See the discussion of combinatory arithmetic in § C.
21. Although the Church theory contained combinators, yet it was not quite combinatorially complete. Functions constant over the whole range of its entities could not be constructed in it; furthermore it lacked a principle of extensionality. We shall say that the Church system was *combinatorially complete in the weak sense.*
22. The title to Rosser's thesis ("A mathematical logic without variables") discloses an anomaly about his system which requires comment. This title seems strange in view of the fact that a system without variables had already been published. The explanation of this anomaly is that Rosser did at one time conceive that he was the first to exclude variables in a certain peculiar sense. Thus in a manuscript copy of his thesis there is a passage, which did not appear in the printed version, to the effect that Curry used the free variable because the latter could prove $x = x$ without any restriction on x. However, in the language used below, the theorem in question is not an elementary theorem, but an epitheorem (cf. Chapter **2**); in the sense of § **2**C Curry's system excludes variables just as rigidly as Rosser's. The point of Rosser's exclusion is that he conceived equality as holding only between entities which were in some sense significant; whereas Curry [GKL: pp. 515 f] maintained that there is no such thing as an insignificant entity, that any combination of the primitives has a meaning, and that questions of significance in a more restricted sense belong to illative combinatory logic. Thus Rosser's exclusion is a relic of the fact that its prototype, the Church system, was an illative system. The discovery of the Kleene-Rosser paradox (see the next paragraph), which occurred while Rosser's thesis was in press, caused large sections of the original manuscript to be discarded, and deprived this particular feature of his work of most of its importance. However there is a vestige of the same point of view in Church's insistence on the possession of a normal form as a criterion of significance, a criterion which reminds one of Behmann [WLM]. Cf. § **3**S3.

jointly, of an inconsistency in the original system of Church.[23] This inconsistency is a theorem of illative combinatory logic. It does not apply to pure combinatory logic, for which the consistency of even the strongest system had been demonstrated in [GKL] (and later was shown more elegantly by methods which we shall study in Chapter **4**). It applied, however, to the strongest of the systems proposed in [PEI].[24] It showed that illative combinatory logic was not so simple a matter as we had naively supposed. A study of the paradox, made in [PKR] and [IFL] with respect to the strongest underlying system of pure combinatory logic, disclosed that the root of the paradox lay in a fundamental incompatibility between combinatorial completeness and the kind of completeness which is expressed by the deduction theorem. Thus a system of illative combinatory logic must either be nonclassical (like Ackermann [WFL], for example) or must formulate in some way the category of propositions.

The foregoing account of the history of combinatory logic contains enough to form a background for the systematic development. Further details will be found at the end of certain chapters below.

One point, however, must be mentioned in passing. Considerations of a combinatorial nature have entered into other systems of mathematical logic. Feys has pointed out, in [PBF], that ideas of this nature occurred in Peano and Burali-Forti. Such considerations are especially prominent in the systems of abstract set theory developed by Fraenkel, von Neumann, Gödel, Bernays, and others.[25] The von Neumann theory even postulates certain functions which are essentially combinators,[26] and perhaps half of his theory is primarily combinatory in nature. These developments,

23. Kleene and Rosser [IFL]. Although very brief, this paper depended essentially on practically all the papers written by both Kleene and Rosser up to that time.

24. It is sometimes stated that the Kleene-Rosser paradox showed the inconsistency of "the system of Curry" as well as that of Church. Thus Rosenbloom [EML, p. 111] says that Kleene and Rosser "found a serious inconsistency in the systems originally proposed by Curry and Church". If this statement is taken literally it is not quite correct. The original system of Curry (i.e. that in [GKL]) was a system of pure combinatory logic and was demonstrably consistent. Curry's further axioms were introduced piecemeal, and the paradox applies only to the strongest of his systems, which had scarcely been applied in practice (cf. § **8**S1). As far as Church is concerned, it happened that he proposed an axiom system for both pure and illative combinatory logic—in fact for the whole of logic—all at once, and this system was indeed shown to be inconsistent. But the part of his system which formulated pure combinatory logic was also demonstrably consistent, so that Rosenbloom's statement, although literally true as regards Church, is hardly fair. It is clear that Rosenbloom did not intend his statement to be taken in this literal fashion, for he himself makes more adequate statements later (l.c., pp. 125 ff).

25. See for example Skolem [BAB]; Fraenkel [UGL], [ATG], [ATW]; von Neumann [AML]; Ackermann [MTB]; Bernays [SAS]; Gödel [CAC].

26. Thus the h of von Neumann's axiom scheme II 6 is the combinator S, and

however, do not use the Schönfinkel application operation as a means of reducing many-place functions to one-place ones, but instead they employ an additional operation of ordered pair. They have until recently been without influence on the course of combinatory logic, and vice versa.[27] However, E. J. Cogan in his doctoral thesis [FTS] showed that the Gödel formulation of set theory can be based quite elegantly upon a system of illative combinatory logic; and that from this point of view about half of the axioms become trivial.

Another development which is related to combinatory logic, but entirely independent of it, is the study of variables which has been made by Menger (see the papers listed under his name in the Bibliography). His work is especially valuable for the analysis of the different uses of variables in ordinary mathematics and physics.

the h of II 7 is the combinator B. Other combinators are implicit in other axioms of his Group II.

27. Note that it is characteristic of combinatory logic as developed here to emphasize the notion of function, whereas set theory in the tradition of Zermelo tends to emphasize the notion of set. Von Neumann's theory is, in this respect, intermediate between combinatory logic and traditional set theory. The more recent theories of Bernays and Gödel tend to revert to tradition, and so to widen the gap between set theory and combinatory logic.