CHAPTER 1

Formal Systems

As a prelude to the study of combinatory logic we shall present in this chapter and the next some ideas about deductive systems in general. Such a system is conceived here as a *formal system*, which is defined as a body of theorems generated by objective rules and concerning unspecified objects. The first section (§ A) contains a rough statement of the approach to this notion and a particular example. A more precise and detailed definition is given in § B. The discussion of §§ C–D is intended to shed light on the nature and significance of the notion. Certain specializations, and the reduction of formal systems to special forms, are considered in § E. There we introduce the ultimate refinement of the notion, viz. the completely formal system. References to other treatments of formal systems, which amplify the text in various ways, and some historical comments are given in § S.

A. THE APPROACH TO FORMAL SYSTEMS

1. Axiomatic systems

To get a first idea of a formal system we start with elementary geometry as taught, after the pattern of Euclid's Elements, in secondary schools.

Elementary geometry begins with certain primitive statements, called *axioms*, which are admitted without proof. From these axioms all other accepted statements are deduced according to logical rules assumed without discussion. The *theorems* are the axioms and the statements deduced from them.[1]

The statements considered in the theory have to do with certain concepts. Some of those concepts are not defined. Others are defined as constructions from the primitive concepts.

In such a concrete axiomatic theory the sense of the terms used and the truth of the axioms is assumed to be intuitively certain. If primitive concepts are left undefined, that is because they are supposed to be intuitively clear; if primitive statements or axioms are left undemonstrated, that is because they are supposed to be intuitively evident. And the theorems derived from the axioms partake of their intuitive evidence.

1. Thus the axioms are among the theorems. We use the word 'theorem' in this sense consistently. When we wish to emphasize that a theorem is not an axiom we speak of a "derived theorem".

As is well known, such concrete deductive theories have been superseded by "pure" deductive theories. Here undefined terms are never tied to an interpretation—it may be they have no known interpretation—they may be treated as designating quite arbitrary things. Undemonstrated statements claim no evidence, as they do not even have presupposed intuitive meanings; they are assumed quite arbitrarily, and the theorems derived from them partake of their arbitrary character. A theory of this character we shall call an *abstract* (or *pure*) *axiomatic system*.[2]

2. Transition to formal systems

Even in such a pure axiomatic theory (under which form most mathematical theories are presented) there remains a naive element, inasmuch as the theory is formalized in terms of logical concepts supposed to be intuitively clear, and the deductions are made by virtue of logical rules whose validity is supposed to be intuitively evident. If we remove this last naive element we arrive at what we call a *formal system*. In such a system the deduction proceeds by arbitrary, but explicitly stated, rules.

We shall define this notion more precisely later on. As already stated a formal system is essentially a set of theorems generated by precise rules and concerning unspecified objects. The perception of the validity of a statement in such a system does not require any experience in the ordinary sense, nor does it require any a priori principles, not even those of logic; it requires simply that we be able to understand symbols employed in a precise way, as we use them in mathematics. A system of this character can be applied without circularity to the study of logic itself.

The statements which the formal system formulates we shall call its *elementary statements*, those which it asserts its *elementary theorems*. The elementary statements (and hence the theorems) are about unspecified objects, which we call the *obs*[3] of the formal system.

3. Example of a formal system

Let us consider a very simple example of a theory, which we shall call the *elementary theory of numerals*, abbreviated as $\mathscr{N}_0$. The obs of

2. The term 'abstract' is used to distinguish such a theory from a naive or concrete deductive theory; it may be omitted when it is not important to emphasize this contrast. In connection with formal systems, 'abstract' is defined slightly differently in § C1. In a broad sense 'axiomatic system' may be understood as including formal systems; but without indication to the contrary an axiomatic system is one in which logic is taken for granted.

3. Previous to 1950 the word 'term' was used in this sense. But this usage was found to conflict with the ordinary use of the same word. On the nature of the obs, see § C2.

this elementary theory will be $0, 0', 0'' \ldots$; i.e., zero, the successor of zero, the successor of the successor of zero, etc.... Elementary statements will be equations between the obs, e.g. $0 = 0$, $0' = 0''$. We take as axiom $0 = 0$, and as rule of derivation "If two obs are equal, their successors are equal." We can then derive elementary theorems such as $0' = 0'$, $0'' = 0''$.

Let us now state this theory more formally. We have to consider:

a. Obs (objects).
  (1) One primitive ob: 0.
  (2) One unary operation, indicated by priming.
  (3) One formation rule of obs: If x is an ob, then x' is an ob.

b. Elementary statements.
  (1) One binary predicate: $=$.
  (2) One formation rule of elementary statements:
  If x and y are obs, then $x = y$ is an elementary statement.

c. Elementary theorems.
  (1) One axiom: $0 = 0$.
  (2) One rule of deduction: If $x = y$, then $x' = y'$.

These conventions constitute the definition of the theory as a formal system in the above sense.

The elementary theorems of this system are precisely those in the list:

$$0 = 0,$$
$$0' = 0',$$
$$0'' = 0'',$$
$$\cdot \quad \cdot \quad \cdot$$

These are true statements about the system. But once the system has been defined, we can make other statements about it, e.g. the statement

If y is an ob, then $y = y$

is a true statement about the system, although not an elementary theorem. That is an example of what we shall call (in Chapter **2**) an *epitheorem*.

B. DEFINITION OF A FORMAL SYSTEM

In this section we generalize the discussion of the preceding example [4] to give a comprehensive definition of a formal system. The definition does not tell us what a formal system is, in the philosophical sense, but describes the nature of the conventions made in setting one up.

4. For other, less trivial examples see [OFP] Chapter V and [APM] § 3.

1. Fundamental definitions

A formal system is defined by a set of conventions which we call its *primitive frame*. This frame has three parts, which specify respectively: (a) a set of objects, which we call *obs*, (b) a set of statements, which are called *elementary statements*, concerning these obs, (c) the set of those elementary statements which are true, constituting the *elementary theorems*.

In its first part, concerning the obs, the primitive frame enumerates certain *primitive obs* or *atoms*, and certain *(primitive)* [5] *operations*, each of which is a mode of combining a finite sequence of obs to form a new ob. It also states rules according to which further obs are to be constructed from the atoms by the operations. It is then understood that the obs of the system are precisely those formed from the atoms by the operations according to the rules; furthermore that obs constructed by different processes are distinct as obs. [6]

In its second part, concerning the elementary statements, the primitive frame enumerates certain *(primitive)* [7] *predicates* each of which is a mode of forming a statement from a finite sequence of obs. It also states the rules according to which elementary statements are formed from the obs by these predicates. It is then understood that the elementary statements are precisely those so formed.

Since the first two parts of the primitive frame have features in common, it is often convenient to treat them together, and to adopt terminology which can be applied to either. Thus the considerations based on the two parts together constitute the *morphology* of the system; the rules of the morphology constitute the *formation rules*; and the atoms, operations, and predicates, taken collectively, constitute the *primitive ideas*. The morphological part of the primitive frame then enumerates the primitive ideas and enunciates the formation rules. To consider simultaneously the properties of operations and predicates we group them together as *functives*. [8] Thus each functive has a certain finite number of arguments; this number will be called its *degree*. As usual, functives of degree one will be called *unary*, [9] those of degree two *binary*, and so on. Given an n-ary functive, the ob or statement formed from n obs by that functive will be called a *closure* of that functive (with respect to those obs as arguments).

5. The word 'primitive' will be dropped, unless it is needed for emphasis or clarity.
6. With reference to what constitute distinct processes, see § **2**B1.
7. See footnote 5.
8. This usage is a little different from that proposed for the same term in [TFD]. However, it is proposed (§ D2) that the latter usage be abandoned.
9. On the reasons for preferring this term to 'singulary' see [DSR] footnote (3) on page 252. See also [rev C].

Occasionally it is expedient to admit among the functives, as predicates of degree zero, certain unanalyzed *primitive statements*. The analogous procedure whereby the atoms are regarded as operations of degree zero will be regarded as an extension of the term 'operation'; unless otherwise stated operations will have positive degree.

The third part of the primitive frame states the *axioms* and *deductive rules* of the system. Axioms are elementary statements stated to be true unconditionally. There may be a finite list of these, or they may be given by rules determining an infinite number in an effective manner (e.g. by axiom schemes). The deductive rules state how theorems are to be derived from the axioms. The *elementary theorems* are the axioms together with the elementary statements derived from them according to the deductive rules. In contradistinction to the morphology, considerations depending essentially on the third part of the primitive frame will be called *theoretical*; taken collectively, they constitute the *theory proper*.

The axioms, deductive rules, and rules of formation together constitute the *postulates* of the system. Thus the primitive frame enumerates the primitive ideas (in its morphological part) and enunciates the postulates.

The foregoing definitions are to be construed as admitting certain degenerate cases. For instance, it is permissible to have a system without any elementary statements. In such a case we should have a system of obs generated from the atoms by the operations; we can study such a system and make statements about it in the manner explained in Chapter **2**. Thus in the example of § A3 we can say that $a = b$ is true when and only when a and b are the same ob, instead of saying it is true when it can be derived from the primitive frame of $\mathcal{N}_0$. Again, a finite set of primitive statements would constitute a trivial formal system without any obs. The admission of these degenerate cases has advantages similar to those of analogous procedures in ordinary mathematics.

In more complex formal systems the enunciation of the rules may require predicates, operations, etc. which do not appear in the elementary statements. Such notions will be called *auxiliary*.[10] Thus it is frequently necessary to divide the obs into categories or types, to define a substitution operation, etc. When auxiliary notions occur the notions necessary for the construction of the elementary statements will be called *proper*. Auxiliary notions are necessary in order to bring certain commonly occurring systems under the concept of formal system. They do, however, introduce a certain vagueness. In the most rigorous conception, that of a completely formal system (§ E5), they are excluded altogether.

The notion of formal system has many analogies with that of an abstract

10. Illustrations of such notions may be found in the examples of [OFP].

algebra in ordinary mathematics.[11] It is therefore expedient to emphasize certain differences.[12] In an algebra we start with a set of elements and a set of operations. The elements are conceived as existing beforehand, and the operations as establishing correspondences among them. Given a sequence of n elements, an operation of degree n "assigns" to this sequence one of the elements as a "value". The case $n = 0$ is admitted, the value being then a "fixed element" or "constant". These fixed elements are not, however, analogous to the atoms; because it is the exception, rather than the rule, that all the elements are obtained from the fixed elements by the operations. Moreover, equality is taken for granted; and it often happens that the same element may be obtained by the operations in many different ways. In these respects the conception of a formal system is totally different. What is given beforehand is not a set of elements but the atoms and operations, and the obs are generated from them. Not only are all the obs obtainable from the atoms by the operations, but each distinct process of construction leads by definition to a different ob; so that an ob is, essentially, nothing more nor less than such a process of generation. That the conception of an abstract algebra can be brought under that of a formal system is a result which will emerge in due course.[13] In the meantime the two conceptions must not be confused.

Finally there are certain notions, allied to those we have considered, which we have not admitted. We could have functives which could be applied to an unspecified finite number of arguments; functives of infinite degree; *connectives*, i.e. notions, analogous to functives, which combine statements to form other statements, so that besides the elementary statements we have certain composite ones; etc. Some of these exclusions are trivial; for example a functive of variable degree can be regarded as a set of functives, one for each admissible degree. Others are dictated by considerations which will be introduced later.

2. Definiteness restrictions

In § 1 we have considered what might be called the anatomy of a formal system, and the emphasis has been on the morphological notions. We turn now to consider certain restrictions which are imposed in order that the system have a finitary, constructive character. A certain unavoidable vagueness will be partially clarified by the discussion in §§ C–D, especially § D4.

11. See, for example, Birkhoff [CSA] p. **441**, or [LTh$_2$] p. vii; Hermes [EVT] p. 153. Note that an algebra is an axiomatic system in the sense of § A1.
12. In relation to the following discussion cf. [DSR] §§ 1–2, especially footnote (2) on p. 251.
13. See, for example, the formulation of group theory in [APM], or of the elementary theory of polynomials in [OFP] Chapter V. Cf also § **2**C.

In the foregoing we have had to do with certain notions which, intuitively speaking, are classes. Thus we have the class of atoms and the class of obs; and similarly the elementary statements, the axioms, the various kinds of rules, and the elementary theorems form classes. Some of these classes—the atoms, operations, predicates, axioms, and various kinds of rules—are given classes; while the obs, elementary statements, and elementary theorems are defined.

Leaving aside the given classes for the moment, we observe that each of the defined classes is specified by a definition consisting of three stages, as follows: first, certain initial elements are specified; second, certain procedures for constructing new elements from given elements are described; and third, it is understood that all the elements of the class are obtained from the initial elements by iteration of these procedures. Such a set of specifications is called an *inductive definition*, and a class so defined is called an *inductive class*.[14] The three stages of an inductive definition may be distinguished as the *initial specifications*, the *generating principles*, and the *extremal specifications* respectively. It is not necessary to state the extremal specifications provided it is understood that the definition is an inductive one.

Given an inductive definition of a class $\mathfrak{C}$, if a construction of an entity A from the initial elements by means of the generating principles is known, then it is clear that A belongs to $\mathfrak{C}$. But if an entity A is produced for testing as to its membership in $\mathfrak{C}$, then there may be no finite procedure for deciding the question. Precisely when this possibility is excluded—i.e., when there is a prescribed process which, given any A, will determine effectively whether it belongs to $\mathfrak{C}$ or not—$\mathfrak{C}$ is called a *definite class*. This definition applies to any class $\mathfrak{C}$, whether defined inductively or in some other way; and it also applies to relations and similar concepts in an analogous manner.

The restrictions to be imposed on a formal system are of two sorts: those which concern the morphology and those which concern the theory proper.

So far as the morphology is concerned we require it to be completely definite throughout. That is, the formulation must be such that the notions of atom, operation of degree n, ob, predicate, and elementary statement are all definite. As regards the given classes, this condition will be met if they are finite; but that is not necessary. The case of infinitely many atoms, for instance, causes no trouble if they are taken as the obs of some more fundamental formal system. We suppose that

14. These terms are due to Kleene [IMM] pp. 258 ff. In previous work of Curry 'recursively generated' and, in earlier work, 'recursive', were used for 'inductive' (cf. the preface to [OFP]).

each of these given classes, if infinite, forms an enumerable sequence;[15] this can be interpreted to mean that its members can be generated as the obs of a system like $\mathscr{N}_0$ (cf. § E4).

With reference to the theory proper we do not require that the elementary theorems form a definite class. If we did, the system would be called *decidable*; and, from some points of view, a decidable system is relatively trivial. What we do require is that the idea of demonstration be definite. A demonstration is a scheme for exhibiting the statement to be proved as derived from the axioms according to the rules. Such a scheme will consist of a sequence of elementary statements, the last of which is the statement to be proved; every member of the sequence of statements is either an axiom or a consequence of its predecessors according to a deductive rule. The idea of a demonstration must be definite in the sense that it can be determined objectively whether or not a supposed demonstration is correct. This implies that the class of axioms must be a definite class.[16]

The validity of an elementary theorem in a formal system so conceived is an objective question. It does not depend on the acceptance of any a priori principle whatsoever. If there are philosophical presuppositions to this process, these presuppositions are necessary for human knowledge and human communication of any kind. On account of this objective character, formal systems may be used for the investigation of logic, even the most basic, without circularity.

For purposes other than those of fundamental logical analysis, these definiteness restrictions are sometimes too stringent. There is indeed some utility in considering systems in which they are relaxed somewhat, and in which there are such notions as functives of infinite degree and rules with infinitely many premises. Such systems might be called indefinite systems; but perhaps it would be better simply to call them semiformal. They will not concern us in this book further.

C. PHILOSOPHY OF FORMAL SYSTEMS

The topics treated in this section concern principally the relations of a formal system to certain activities we engage in with respect to it. The

15. This corresponds to the fact that we ordinarily designate such atoms by a letter with numerical subscripts, e.g. $x_1, x_2, \ldots$.

16. Cf. [OFP] pp. 31 ff. Note that it is not necessary that it be a definite question whether or not a given sequence of elementary statements constitutes a demonstration. We may require of a demonstration that the rule to be applied be indicated at each step, that the premises be explicitly indicated (e.g. by exhibiting the proof in the form of a genealogical tree) etc. What is important is that when all the required information is given the correctness of an alleged demonstration be definitely verifiable. (Cf. Kleene's notion of a deduction with an analysis in [IMM] p. 87).

discussion is intended to throw light on the nature and significance of a formal system, and to clarify the definition in § B. One topic belonging in this section, the relation of formal systems to language, is of such complexity and importance that it is reserved for a separate section (§ D).

1. Presentation

A particular enunciation of the primitive frame of a formal system will naturally employ symbols to designate primitive ideas; and it must describe how the closures of the functives are to be symbolized. Such a particular enunciation, with its special choice of symbolism, we call a *presentation* of the primitive frame (and of the formal system).

It is clear that a formal system can be communicated only through a presentation. It is also clear that the particular choice of symbolism does not matter much. So long as we satisfy one indispensable condition—namely that distinct names be assigned to distinct obs [17]—we can chose the symbolism in any way we like without affecting anything essential. We can, therefore, regard a formal system as something independent of this choice, and say that two presentations differing only in the choice of symbolism are presentations of the same formal system. In this sense a formal system is abstract with respect to its presentation.

2. Representation

A presentation of the primitive frame speaks about the obs, using nouns to name them; but it does not specify what determinate thing each ob is. If we assign a unique determinate thing to each ob in such a way that distinct things are always assigned to distinct obs, we have a *representation* of the system. The things assigned may be any kind of objects about which statements can be made: symbols (or symbol combinations), qualities, numbers, ideas, manufactured things, natural beings. The assignment of such a representation (or a change of representation once made) does not affect in any way the criterion of truth for the elementary statements, and thus the predicates remain defined by the primitive frame without reference to any external "meaning". A system does not cease to be formal because a representation is assigned to it; but the representation is, so to speak, accidental, and the system, as such, is independent of it. A system for which no representation is specified will be called *abstract*.

Given any formal system, we can find a representation for it in various ways. Thus we can find a representation in terms of symbols for the elementary theory of numerals as follows: Let 0 be the asterisk ('*'), and

17. This condition would be violated for example, if we admitted a second operation, indicated by double priming, in the system $\mathscr{N}_0$; for then it would be uncertain whether $0'''$ meant $(0')''$ or $(0'')'$.

let the operation of the system be the addition of an exclamation point ('!') on the right. Then the obs of $\mathscr{N}_0$ become the symbols on the right in the following table (on the left are the names of these obs):

0	*
0′	*!
0″	*!!
. . .	

It can be shown that a quite general type of formal system may be represented in the linear series formed from two or more distinct symbols. [18] The predicates are defined solely by the primitive frame and have no reference to any extraneous "meaning".

It follows from what was said in § 1 that the names of the obs in any presentation constitute a representation. We shall call such a representation an *autonymous representation*. It is a way of conceiving a formal system which is close to the ideas of Hilbert. It is free from ambiguity as long as the system is not considered in the same context with an interpretation (see § 3) in which the symbols of the presentation have other meanings. [19] (Cf. § **2**S1 and the discussion at the end of [SFL].)

3. Interpretation [20]

The idea of representation is to be contrasted with that of interpretation. By an *interpretation* of a formal system we mean a correspondence between its elementary statements and certain statements which are significant without reference to the system. Let us call the latter statements *contensive* statements. (We shall use the adjective 'contensive' [21] quite generally as indicating something defined antecedently to the system.) It is understood that one formal system may have an interpretation in another, or it may have a completely intuitive interpretation.

The notion of interpretation does not require that there be a contensive translation for every elementary statement. We have to make allowance for the fact that a formal theory may be an idealization, rather than a

18. See [LLA], Appendix, § 3; Quine [TCP]; Rosenbloom [EML] p. 171. For a representation in terms of manufactured objects, see also [APM], § 4, p. 231. For further discussion of the philosophy see [OFP] Chapter VI. A representation in terms of one symbol is equivalent to a representation in terms of numbers (i.e. obs of $\mathscr{N}_0$); this can always be obtained by the methods of Gödel.

19. Of course we are not committed to this representation in what follows. Our treatment is abstract; and the reader who wants a representation can choose either an autonymous or heteronymous one as he pleases.

20. For an amplification of §§ 3–4 see [OFP] Chapter XI and [TEx]. The topic is treated briefly here because it is not particularly relevant for combinatory logic. But what is said here requires some revision in the cited papers.

21. This word is a translation of the German 'inhaltlich', which is not exactly rendered by the usual translation 'intuitive'. It was introduced in [APM].

mere transcription of experience.[22] Thus if certain physical theories were formalized as formal systems, they would doubtless contain statements not translatable into statements which can be tested experimentally. Still less is it necessary that there be a contensive object corresponding to each ob. Thus in interpretations of systems containing variables (§ 2C) contensive objects are ordinarily not associated with the variables, but only with the constants. Further, when such contensive objects exist, it is not always the case that distinct ones are associated with distinct obs.

A *valid* interpretation is one such that every contensive statement which corresponds to an elementary theorem is true; in that case the system will also be said to be valid for the interpretation. A *direct* interpretation is one in which at the same time a contensive object is assigned to each ob (but not necessarily distinct objects to different obs).

The interpretation of one formal (or semiformal) system in another is important in modern logic. Under certain conditions, too complex to be considered at this stage, this gives rise to "models"

4. Acceptability[23]

We shall consider a few general principles concerning the reasons which lead us to choose a formal system for study. These reasons are relative to some purpose; when they are fulfilled for a given purpose, we say that the formal system is *acceptable* for that purpose.

Naturally the most important criterion for acceptability is the validity of an interpretation in some field we are interested in. In fact we usually have a concrete theory first, and then derive the formal system from it by abstraction. Thus, in setting up the elementary theory of numerals, we specified that 0 was the intuitive zero, that x' was the successor of x and that $=$ was the predicate expressed by 'equals'; then we abstracted from this interpretation. In the more complex theories of science the situation is less trivial but analogous. As already mentioned, a certain amount of idealization may be involved. In such cases the validity, at least approximate, of the interpretation is the sine qua non for acceptability.

There are, however, other criteria. Of two systems equally valid for the same interpretation, one may be simpler, more natural, or more convenient than the other; or one may give a more profound and illuminating analysis; or one may suggest relationships with other fields of study and open up new fields of investigation; or one may conform better to certain philosophical prejudices; etc. Thus acceptability has to be distinguished from validity; the latter is a truth relation between the system as a whole

22. Cf [HB.I], pp. 2–3.
23. See footnote 20.

and the subject matter to which it is applied, the former takes into account our purposes in studying that subject matter.

When a formal system is considered in connection with an application there are two sorts of truth concept to be distinguished. The truth of an elementary theorem of the formal system is determined by the abstract nature of the theory itself. Validity and acceptability are properties of the system as a whole in relation to a subject matter; if the subject matter is empirical, they are empirical also. If the contensive analogue of an elementary theorem is found to be false, that does not affect the truth of the theorem; it simply shows the invalidity of the interpretation.[24] For an empirical subject matter validity can only be determined hypothetically. In such a case a convenient and useful system is held to be acceptable as long as no invalidity is known; when one is discovered the system has to be abandoned or modified.[25]

D. LINGUISTIC ASPECTS OF A FORMAL SYSTEM

In the explanation of a formal system, as in any intellectual activity, language is used; and this language is used, as is any language, to name objects and make statements about them. The formal system is not a discussion about this language. But we can shed light on the nature of a formal system by examining this language according to the practice of modern semiotics.[26] This we shall do in the present section; and at the same time we shall consider the relation between a formal system and the more linguistically oriented notions which are the current fashion.

1. Languages

By a language, in the general sense used in semiotics, is meant any system of objects, called *symbols*, which can be produced in unlimited quantity, like the letters of ordinary print or the phonemes of speech, and combined into linear series[27] called *expressions*. It is irrelevant whether or not the language is used for communicative purposes; if it is so used it will be called a *communicative language*.

In order to talk about a language we need some way of referring to its symbols, expressions, and other parts. For the expressions it has now become standard practice to use a specimen of the expression, enclosed in

24. Of course the discovery of an internal inconsistency would entail invalidity for any interpretation.
25. On the effect of an inconsistency see § 8S3.
26. In the sense of Morris [FTS]. See also § S2 at the end of this chapter.
27. Cf. [LFS] opening paragraph; [TFD] p. 11. We are restricting attention to what are called linear languages in these papers.

quotation marks, as a name of the expression. This usage conflicts to some extent with other uses of quotation marks; we use single quotation marks in order to avoid some of the confusion. Thus Brussels is the capital of Belgium, but 'Brussels' is an eight-letter word which constitutes the name of that city in English.

2. Grammatics

We sometimes wish to consider parts of a communicative language with reference to their grammatical functions. A combination of symbols with a grammatical function will be called a *phrase*. It is clear that not all expressions are phrases; but it is also true, since phrases may consist of detached parts, that not all phrases are expressions. The simplest phrases will sometimes be called *words*.

There are three main kinds of phrases, namely: *nouns*, which name objects; *sentences*, which enunciate statements, and *functors*, which combine phrases to form larger phrases. The functors include *operators*, which form nouns from other nouns; *predicators*, or verbs, which form sentences from nouns; and *connectors*, which form sentences from other sentences. [28] For the meanings of phrases in general we shall sometimes use the words 'concept', 'idea', or 'notion'. The concepts corresponding to the six kinds of phrases just enumerated will be called *objects* (obs, terms), *statements* (propositions), *functions*, *operations*, *predicates*, and *connections* respectively. In each case we shall say that the phrase *designates* its concept; 'designate' does not, like 'name', imply that its subject is a noun, nor does it have a specific philosophical connotation. [29]

The terms 'degree', 'closure', 'unary', 'binary', etc. are defined for functors just as they were for functives in § B1, except that the arguments of a functor may be phrases of any kind. The terminology may also be applied to functions.

These distinctions have reference to a communicative language. But it may happen that these and related ideas are significant in a more abstract situation. The study of the consequences of these conventions has been called *grammatics*, and a language in which they are significant a *grammatical language*.

In order to have a complete notation for referring to functors we should need to indicate how the closure is to be symbolized, and this would necessitate some indication of blanks into which the arguments are to be inserted. But usually functors are of the following three types: *prefixes*, written before the arguments; *infixes*, which are binary functors written between the two arguments; and *suffixes* written after the arguments. Prefixes and suffixes can have any number of arguments, and as Łukasiewicz has shown, parentheses are not necessary when either of these two kinds occurs alone; [30] infixes are necessarily binary and require parentheses. Since the rules of parentheses can be regarded as known, it will be sufficient, in the simpler cases, to treat the functor as a simple symbol; when this symbol is used as a noun, it will be understood to be a name for the corresponding function. Thus, in the notation of § A3, ' ' ' is a suffixed unary operator, ' = ' an infixed (binary) predicator, = is a binary predicate. In ordinary mathematics ' + ' is an infixed operator, + a binary operation. In more complex cases, however, it will be necessary to use a notation involving blanks; thus '$(---_1)^{---_2}$' for exponentiation, '$---_1(---_2)$' for the application of a function to an argument, '--- is positive' for the property of being positive, etc. [31]

28. This is not, of course, intended to be an exhaustive classification.
29. The use of separate names for the phrases and for the meanings of the phrases does not commit one to any special philosophy. Thus one could maintain that there is really only a difference of emphasis between the two uses.

3. The U-language

The construction of a formal system has to be explained in a communicative language understood by both the speaker and the hearer. We call this language the *U-language* (the language being used). It is a language in the habitual sense of the word. It is well determined but not rigidly fixed; new locutions may be introduced in it by way of definition, old locutions may be made more precise, etc. Everything we do depends on the U-language; we can never transcend it; whatever we study we study by means of it. Of course, there is always vagueness inherent in the U-language; but we can, by skillful use, obtain any degree of precision by a process of successive approximation.

4. The A-language

We may consider the formulation of a formal system as an introduction of new symbols into the U-language. In the statement of a primitive frame we use two kinds of symbols: on the one hand words or expressions already known, on the other hand quite new symbols. In the presentation of our elementary system of numerals, words such as 'statement', 'ob', 'operation', 'theorem' are words which are supposed to have a meaning

30. For a proof of this theorem and references to preceding proofs see Rosenbloom [ETM] Chapter IV § 1. Another proof is given in [LLA] Appendix 2. For a recent discussion see Łukasiewicz [FTM] and the remarks attached to it. See also A. W. Burks, D. W. Warren and J. B. Wright in Math. Tables and other Aids to Computation 8:53–57 (1964).
31. The foregoing account of grammatics is abridged from [TFD], Chapter I, § 5. The distinction made there between 'function' and 'functive' is not retained here, as the notation in '-ive' seems unnatural. Thus 'function' here corresponds to a 'functive' there; for 'function' there we shall use the term 'closure of a function', etc. This frees the term 'functive' for a more special purpose in § B1.

in the U-language (which includes the presupposed technical terminology) before the formal system is introduced. Even the variables '*x*' and '*y*' are used in a sense known intuitively in U (therefore they may be called "U-variables"). But symbols such as '0', ' ′ ', ' = ' are new; they are, it may be said, the elementary symbols of a certain new language—in the generalized sense mentioned above—which is embedded in the U-language. We call this language the *A-language.*

More generally, given a certain presentation of a formal system, the A-language is that language (in the generalized sense) which is constituted by the symbols and expressions used for the primitive ideas and their combinations. The symbols of the A-language are adjoined to the U-language to be used there; they perform grammatical functions therein. Thus the symbol '0' in the A-language of $\mathscr{N}_0$ is one of a particular kind of nouns designating what we have called "numerals". The symbol ' ′ ' is a suffix which, if put to the right of one of these nouns, forms from it another of these nouns. The symbol ' = ' is a verb which, when placed between two such nouns, forms from them an *elementary sentence.* [32] The grammatical conventions just enunciated express in fact the same ideas as the morphological part of the primitive frame of $\mathscr{N}_0$. What the primitive frame describes is thus the structure and mode of use of the technical A-language embedded in the U-language. [33]

We can thus regard the process of defining a formal system from a linguistic point of view. If we do this we can sharpen somewhat the provisions of § B. For example, the definiteness restrictions can be stated as follows: given an expression of the language U + A, it must be a definite question whether it is the name of an atom, the name of an ob, a predicator designating a predicate, or an elementary sentence; if it is an elementary sentence it must be definite whether it is an axiom; and, given an application of a rule, it must be definite whether the rule is allowed and whether it is correctly applied.

An overemphasis on this point of view, however, tends to obscure the abstractness of a formal system with respect to its presentations.

5. Syntactical systems

A general formal system as formulated in § B may be compared with the syntactical systems by means of which mathematical logic is often

32. We shall use this term for a sentence expressing an elementary statement.

33. It has been noted that the U-language can change and that the adjunction of the A-language is such a change. For definiteness we shall speak of the U-language taken prior to the adjunction of the A-language as the *original U-language,* and use the expression 'U+ A' to indicate the U-language with the A-language adjoined. In most cases, however, the simple term 'U-language' will suffice. For the origin of the terms 'U-language' and 'A-language', see [LFS] and [LMF].

described. In these syntactical systems we have two distinct languages: the U-language, and an *object language* which is kept entirely distinct from the U-language, in the sense that its symbols are talked about, but never used. The objects of discussion are expressions of this object language. [34] If all the expressions of the object language enter in the theory, it will be called a *complete syntax; a partial syntax* deals only with certain *well-formed*[35] expressions.

The only difference between a formal system and a syntactical system lies in the description of the objects; one can formulate elementary statements and theorems (and so on to epitheorems) for the latter system much as for the former. The representation mentioned in § C2 converts an arbitrary formal system into a partial syntax of a suitable object language. Conversely, if we associate with each object symbol a unary operation, viz. that of prefixing it, and take as atoms either the empty expression or all the expressions consisting of a single symbol standing alone, then a complete syntax will be exhibited as a formal system represented in the object expressions, and a partial syntax can be treated by introducing the well-formed expressions as an auxiliary category. In such a case concatenation is defined recursively, much as addition in Peano arithmetic.[36] If, however, concatenation is taken as a primitive operation, then some additional formalization will be necessary to make the system strictly formal; and it will have a direct interpretation, rather than a representation, in terms of the object expressions.[37] In either case the system becomes an abstract system by simply deleting the object language.

Syntactical and abstract systems each have advantages, but one may be as formal as the other. A syntactical formulation entails certain complications, [38] and it is in a sense less abstract than the nonsyntactical

34. We understand 'syntactical system' as denoting any theory dealing with the expressions of the language so as to take account solely of the kind and arrangement of the symbols of which they are composed. The theory may be complete or partial; also it is immaterial whether some category of expressions is singled out as "sentences", and if so whether a relation of direct consequence is defined. Systems described in the literature as "semantical" may be syntactical in this sense. Cf. § S2.

35. This term, introduced in Church [SPF. I], is finding wide acceptance.

36. These ideas are largely due to Hermes [Smt]. For recursive definitions see § 2E. A formalization of equality (see the next sentence) may still be necessary in order to express the rules. See also Tarski [BBO], [WBF] and Rosenbloom [EML] pp. 189 ff.

37. Since concatenation is associative, the same expression may sometimes be constructed in different ways and hence may correspond to different obs. For elaboration of the point made in the text see [LLA] Chapter I, § 5 (cf. [SFL]); on the formalization see the references at the end of § S2.

38. For the discussion of these complications see also [OFP] pp. 41–46, [LLA] I § 5, [SFL].

formulation. We shall formulate the different forms of combinatory logic as abstract systems.[39]

The fact that we are dealing with an abstract, rather than a syntactical, system has a consequence of some importance. Writers on semiotics have pointed out the dangers of the "autonymous mode of speech"—viz., that in which a specimen of a symbolic expression is used as name for that expression. These dangers are particularly serious when the subject matter is linguistic; one requires not only quotation marks, but further devices (e.g. Quine's "corners") to be sure of avoiding the "confusion between use and mention". But when the subject matter is nonlinguistic, or when the object language is uninterpreted (cf § C2), these dangers are not so great; the devices of ordinary language are usually sufficient to make the meaning clear. We have therefore permitted ourselves, on occasion, a certain looseness in regard to these matters; and beyond single quotation marks we have not introduced any technical notation for the purpose. (Cf [OFP] pp. 44 ff; Carnap [LSL] pp. 156 ff; Quine [$ML g_2$] §§ 4, 53.)

E. SPECIAL FORMS OF FORMAL SYSTEMS

Formal systems have been defined in § B with such generality that any rigorous deductive theory can be put in the form of a formal system. In this section we consider formal systems of special forms and the reduction of more general formal systems to them.

1. Relational systems

A formal system in which there is a single primitive predicate and that a binary one we shall call a *relational system*. Such a system in which the primitive predicate has the properties of equality—reflexiveness, symmetry, transitiveness, and certain properties of replacement which we shall discuss in Chapter **2**—will be called an *equational system*; one in which the primitive relation is a quasi-ordering (see § **2**D) a *quasi-ordered system*.

In most systems of ordinary mathematics equality is taken for granted. Equational systems frequently arise in a natural manner when such a system is formalized. The system $\mathscr{N}_0$ is an equational system. Less trivial examples are the formulation of group theory in [APM] § 3 [40] and various formulations of Boolean algebra. Boolean algebras, lattices, etc. can also be formalized naturally as quasi-ordered systems.[41]

It can be shown that an arbitrary formal system can be reduced to a

39. In this discussion we have avoided the term 'metalanguage'. For this see § S2.
40. Repeated in [TFD] I § 2 and [LLA] II § 2.
41. See, e.g., [LLA].

relational one.[42] But there seems to be no point in this unless the system obtained is equational (or quasi-ordered) and of such structure that the analogy with ordinary algebra is of some importance.[43]

We shall find it convenient to obtain an equational system of the theory of combinators as an intermediate stage in Chapter **6**. The reduction of this by the method of § 2 will concern us in Chapter **7**.

2. Logistic systems

A formal system in which the only predicate is a single unary one is called a *logistic system*.

In such a system the single predicate represents a class of obs, and each elementary statement is to the effect that some ob belongs to that class. In this book we shall call the obs of that class *asserted obs*, and we shall say that a statement asserts X just when it says that X belongs to that class. We employ the Frege sign '$\vdash$' as prefix to designate the predicate. Hilbert's 'ist beweisbar' and Huntington's 'is in T' are other ways of expressing it; in the literature it is often not expressed at all, but left to the context.

We now show that *an arbitrary formal system is in principle reducible to a logistic one with an additional auxiliary category*. In fact let $\mathfrak{S}$ be the given formal system, and let us choose a presentation for it in which '$\vdash$' is not used. The logistic system to be constructed we call $\mathfrak{S}'$. We allow two kinds of obs in $\mathfrak{S}'$, *basic obs* and *formulas* (or propositions);[44] the basic obs will coincide with the obs of $\mathfrak{S}$, and the formulas will correspond to the elementary statements of $\mathfrak{S}$. We can evidently achieve the first of these ends by using the same atoms and operators in $\mathfrak{S}'$ as in $\mathfrak{S}$, and translating[45] all the ob-formation rules from obs of $\mathfrak{S}$ to basic obs of $\mathfrak{S}'$. To each predicate of $\mathfrak{S}$ let there correspond an operator of the same degree in $\mathfrak{S}'$, and let this correspondence be one-to-one. Let the formation rules be such that whenever the predicate forms an elementary statement $\mathfrak{A}$ in $\mathfrak{S}$, the operator forms a formula A in $\mathfrak{S}'$; the correspondence between $\mathfrak{A}$ and A is also one-to-one. Let the predicate $\vdash$ form an elementary statement of $\mathfrak{S}'$ from an arbitrary formula. Let the statement

$$\vdash A$$

in $\mathfrak{S}'$ be taken as the translation of $\mathfrak{A}$. In this way the primitive frame

42. This is a trivial consequence of § 2. For we can replace '$\vdash A$' by '$A \; \boldsymbol{R} \; 1$', where 1 is a new atom.
43. For a study of such transformations in connection with propositional algebra see [LLA] Chapter II.
44. The term 'proposition' is allowable here since we have not used it in another sense. That would agree with [IFI], where it is proposed to use it henceforth for a category of obs. However, the term 'formula' is more usual.
45. If there are several categories of obs in $\mathfrak{S}$, there will be corresponding categories of basic obs in $\mathfrak{S}'$.

for $\mathfrak{S}$ can be translated into one for $\mathfrak{S}'$. The elementary theorems of $\mathfrak{S}'$ are precisely the translations of those for $\mathfrak{S}$, and the relation between the two systems has a character similar to that between two presentations of the same formal system.

To illustrate this process we apply it to the system $\mathscr{N}_0$. Let an infixed '$\square$' [46] indicate the operation corresponding to equality. Then if $\mathfrak{S}$ is the system $\mathscr{N}_0$, the primitive frame for $\mathfrak{S}'$ will be as follows:

0 is a basic ob.
If a is a basic ob, then a' is a basic ob.
If a and b are basic obs, $a \square b$ is a formula.
If A is a formula, $\vdash A$ is an elementary statement.
$\vdash 0 \square 0$.
If a and b are basic obs and $\vdash a \square b$, then $\vdash a' \square b'$.

In this case it is clear that the extra auxiliary category is unnecessary; for if basic obs and formulas were merged into one category of obs, then there would be some strange obs, e.g. $(0 \square 0')'$, but no change in the elementary theorems. This cannot be affirmed in general.

This process applies also to systems allowing connectives, which were excluded at the end of § B1. In fact, we can let a connective in $\mathfrak{S}$ correspond to an operator in $\mathfrak{S}'$ which combines formulas to give other formulas. Various other generalizations might be included.

It follows from this that one can, without loss of generality, confine attention to logistic systems. In modern logic this is done almost universally. In that case it is natural to have a different conception of some notions connected with formal systems. This difference can be illustrated in regard to the term 'axiom'. In § B an axiom is an elementary statement; it is expressed by a sentence in the U-language. In a logistic system this statement asserts a certain ob (formula). It is quite common to apply the term 'axiom' to the ob (formula) rather than to the statement which asserts it. Analogous differences pertain to some other notions. [47]

3. Applicative systems

By a method essentially due to Schönfinkel [BSM] we can accomplish a reduction to a system with a single binary operation. In fact, let there be introduced into the system new atoms corresponding to the original operations. We adjoin a binary operation and denote it by simple juxtaposition. To avoid superfluous parentheses, let it be understood that in a row of symbols without parentheses the operations are to be performed

46. Read "quad".
47. Cf. [OFP] p. 35; [IFI]. In the latter axiomatic statements and axiomatic propositions (i.e. formulas) are distinguished, and 'axiom' can mean either, according to the context.

from left to right, so that '$abcd$' for instance, will mean the same as '$((ab)c)d$'. Let F be the ob associated with an m-ary operation f. Then we can define $f(a_1, a_2, \ldots, a_m)$ as $Fa_1a_2\ldots a_m$, and so eliminate f as a primitive.

For example, suppose a system contains an operation of addition; if A is the associated ob, we can define $a + b$ as Aab. In our system $\mathscr{N}_0$ put into logistic form, we may associate an ob S with the operation $'$ and an ob Q with the operation $\square$; then we can define x' as Sx and $x \square y$ as Qxy, eliminating $'$ and $\square$ as primitives.

The new operation is called *application*, because in the case $m = 1$ it corresponds to the application of a function to an argument. A study of its intuitive meaning in other cases will be made later, in Chapter **3**. Of course it allows the possibility of forming intuitively nonsensical combinations (like AA); but these will not concern us here.

A system with application as sole operation will be said to be an *applicative system*; if it has application in combination with other operation(s) it will be said to be *quasi-applicative*. Then any system can be reduced to an applicative system, if we introduce additional obs.

For most purposes in this book we shall use the notation just described for the application operation. There is, however, an alternative notation which avoids parentheses altogether. This is obtained, according to the result of Łukasiewicz mentioned in § B2, [48] by using a binary prefix for an operator. A suitable such operator is the '$*$' of Chwistek. [49] Then '$fa(gb)(hcd)$' translates into '$***fa*gb**hcd$'. The notation can be improved by using exponents to indicate repetitions of '$*$'; the above then translates further into '$*^3fa*gb*^2hcd$'; and '$Fa_1\ldots a_m$' into '$*^mFa_1\ldots a_m$'.

4. Reduction to a single atom

We may reduce any formal system to a system with only one primitive ob. [50] For suppose the system is applicative and has the primitive obs E_1, E_2, E_3, ... Let C be a new primitive ob, and define (where '$\equiv$' indicates identity by definition):

$$E_1 \equiv CC,\ E_2 \equiv CE_1,\ E_3 \equiv CE_2, \ldots$$

Then the resulting system has essentially the same content as the old one and has only one primitive ob C.

This is evidently a purely technical reduction; from the standpoint of intelligibility it is rather a disadvantage. However, essentially the same idea has been used by Post, Chwistek, and others [51] to reduce an infinite

48. Footnote 30.
49. [GNk] p. 62 (= [LSc], p. 86); cf. Chwistek and Hetper [NFF]. Rosenbloom [EML] p. 111 uses a vertical stroke '|'. In [OFP] 'α' was used.
50. This reduction was also given by Schönfinkel [BSM].
51. See, e.g., Rosenbloom [EML] pp. 40 ff, 158 ff.

number of primitive obs to a finite number. Thus instead of the sequence

$$x_1, x_2, x_3, \ldots$$

we can introduce the primitive obs X, S (where the S can be an ob already in the system), and define

$$x_1 \equiv X,$$
$$x_2 \equiv X' \equiv SX,$$
$$x_3 \equiv X'' \equiv S(SX),$$
$$\cdots\cdots\cdots$$

If there is a second sequence

$$y_1, y_2, \ldots,$$

we can use a primitive ob Y in combination with the same S or the same X in connection with a different S. One can proceed somewhat similarly with double, triple sequences. It is evident that all this amounts to formalizing the numerical subscripts by incorporating in the system the morphology of the system $\mathcal{N}_0$. Thus there is a sense in which that system is contained in every infinite system if it is sufficiently formalized.

5. Complete formalization

A system will be called completely formalized just when it contains no auxiliary notions and no restrictions on the applicability of its functives. There will then be only one category of obs; the closure of an n-ary operator with respect to any n obs will always be an ob; and that of an n-ary predicate with respect to any n obs will always be an elementary statement. These stipulations being understood, its morphology will consist simply of an enumeration of its primitive ideas. The primitive frame can then be organized as follows:

(a) Primitive ideas.
  (1) Primitive obs (atoms).
  (2) Operations (classified as to degree).
  (3) Predicates (classified as to degree).
(b) Postulates.
  (1) Axioms.
  (2) Deductive rules.

This formulation removes some of the vagueness which remains inherent in the definition of § B, since morphology is formalized as well as theory. It might be taken as the basic conception; the notion of incompletely formal system is a concession to practical needs. [52]

The system $\mathcal{N}_0$ is completely formalized, as its morphology consists

52. Cf. [LLA] p. 24; [OFP] p. 36 f.

simply of an enumeration of the primitive obs (the single ob 0) and of the primitive operations and predicate, viz., ′(unary) and = (binary).

Examples of incompletely formalized formal systems would be: (1) a system whose morphology divides the obs into several "types", (2) a system with a rule of substitution, as this rule has to specify a peculiar class of obs upon which the substitution may be performed. [53]

It is plausible that any formal system can be reduced to a completely formalized one by introducing additional primitive ideas, so that the morphological and auxiliary notions can be expressed in the A-language (their rules being transferred to the theoretical part of the primitive frame).

The reduction of certain general types of formal systems to completely formalized ones is one of the tasks of combinatory logic. We shall see that it can be accomplished in such a way as to lead to systems of strictly finite structure, and at the same time to preserve naturalness in a way which the artificial reductions of § 4 do not.

S. SUPPLEMENTARY TOPICS

1. Historical and bibliographical comment

The history of formal methods in general lies outside the province of the present work. We confine ourselves here to citing those works which have had a traceable direct effect on the development of our ideas, to mentioning certain historical and expository material which we have noticed and appears to us to deserve some emphasis, and to indicating where the treatment in the text may be amplified in certain directions. Further information on the history may be obtained from the histories of mathematics and of logic. The bibliographic citations may be supplemented by the general bibliographies in [CB], Fraenkel [AST]. Briefer but useful bibliographies are found in Wilder [IFM], Kleene [IMM], Beth [SLG], and Church [BBF]. The latter two are especially helpful for the history of logic.

For general treatment of axiomatic methods we refer to: Hilbert [ADn], [GLG_7]; [HB], § 1; Weyl [PMN]; Wilder [IFM], Chapters I–II (contains references); Tarski [ILM], Chapter VI; Kleene [IMM], especially pp. 59–72, 246–251; Young [LFC]; Bell [MQS]; Huntington [CTS]; Kershner and Wilcox [AMth]; Dubislav [PMG]; Fraenkel [EML_3] Kap. 5 (a revision, to appear in English, has been announced); Cavaillès [MAF]. There are in addition several popular books which we have not examined.

The notion of axiomatic system was evolved during the nineteenth century from the idea of interpreted deductive system as described in § A1. The movement was especially important in geometry, where it culminated in the formulations of Pasch, Hilbert, Pieri, Veblen, and others. For the history of this movement we refer to Nagel [FMC]; Enriques [HDL]; Jørgensen [TFL]; Cavaillès [MAF] Chapter I. Briefer accounts are found in the general histories of Bell [DMth] and Cajori [HMth]; and there is some mention of the history in Wilder [IFM] and Kleene [IMM].

During the early years of the twentieth century this movement was continued in America in a series of memoirs on the applications of the axiomatic method to various branches of mathematics. Some samples of these papers are: Dickson [DGF]; Huntington [CTS], [FPA], and various papers cited by Fraenkel [AST]; Veblen [DTO], [SAG], [SRR]. The excellent exposition of Young [LFC] appeared in this

53. For examples of these eventualities see Examples 4, 5, 6, 7 of [OFP] Chapter V.

period; cf. also the introduction to Veblen and Young [PGm.I]. The general analysis of E. H. Moore [IFG], which was expressed in a symbolism showing strongly the influence of Peano, used postulational and epitheoretical methods extensively. Naturally there were similar developments in other countries; but they did not have the same direct influence on the present work.

Up to the present we have been considering the notion of axiomatic system, i.e., one taking logic for granted. But one can hardly formulate the notion of an axiomatic system without asking almost immediately how logic is to be formulated. (For a fairly recent example of this transition see Bôcher [FCM].) Consequently the two types of formalization developed almost simultaneously. One finds a surprisingly modern conception of formalization already in Boole [MAL]; and a strictly modern notion of syntactical system occurs in Frege (see Scholz [WIK]). We shall not go into a remote history of this development here, but refer the reader to standard treatments of the history of logic. This remoter history did not have a direct influence on the present work.

The Principia Mathematica [PM], which appeared in 1910, was somewhat deficient in the strictness of its formalization (see § 0A); but it exerted a great influence on the development of mathematical logic generally. It was preceded by various other papers by the authors (for citations see [CB]). It more or less dominated the subject in England and America during the first third of the twentieth century.

During this period of dominance the notion of axiomatic system, and formal reasoning generally, was discussed philosophically by various persons. The following may be cited as having had an influence at one time or another on the present work: Dodgson [54] [WTS]; Brouwer [IFr]; Lewis [SSL] Chapter VI; Keyser [MPh]; Eaton [STr] Chapter VII; Church [AZA]; Dresden [PAM]; Pierpont [MRP]; Weiss [NSs]; Dubs [NRD]. The work of Sheffer, although highly regarded by his pupils, had little direct influence because of its inaccessibility. [55] Particular attention should be paid to Lewis l.c.; he obtains a syntactical formulation which is strictly formal, but apparently quite independent of Hilbert.

In the twentieth century the development of formal methods on the continent of Europe was dominated by the Hilbert School and its altercation with the intuitionists. The development of Hilbert's ideas is summarized (with references) in Bernays [HUG]; cf. also Weyl [DHM], Cavaillès [MAF]. A systematic exposition, which has become a classic, is given in [HB]. For the details of the evolution one should consult the papers by Hilbert, Ackermann, Bernays, and von Neumann listed in the general bibliographies. We cite as particularly important for the present work: Hilbert [ADn], [NBM], [Und]; Bernays [PMth], [PEM]; and von Neumann [HBT]. The dispute with the intuitionists is no doubt responsible for sharpening some of the ideas; for claims of the latter in this connection see Brouwer [IBF], Heyting [MGL]. For an intelligible account of the ideas of intuitionism see Wilder [IFM] Chapter X; cf. also Heyting [MGL]. There is an extensive critical literature of the Hilbert-Brouwer dispute; of this we cite Hardy [MPr], Fraenkel [EML_3], § 18, Heyting [MGL], and the general references on formal methods given above. Hilbertism developed into the syntactical conception of formal methods, which is currently the most fashionable (see § 2).

The notion of formal system expounded in this work is that developed in a series of papers by Curry. The principal papers in this series, in the order of composition (which is quite different from that of publication) are: [ALS], [OFP], [APM], [TFD] Chapter I, [LLA] Chapter I; briefer summaries are given in [RDN], [PKR] § 2, [SFL]; while special topics are dealt with in [MSL], [FRA], [LFS], [LMF], [LSF], [DSR], and [TEx]. Of these papers [ALS] is now obsolete; the author was thinking

54. This precedes the period, but belongs with it in spirit; and became known to us rather later than the others.

55. Curry recalls having seen a copy of Sheffer [GTN] in the library of Harvard University, but has almost no recollection of its content.

in conceptualistic terms which he later regarded as irrelevant; and he had been hardly influenced at that time except by the Anglo-American stream in the above account. The booklet [OFP] is the most extensive of the series; although published only recently, most of it was written in 1939. [56] The later papers were mostly condensations of parts of [OFP] with modifications and revisions (which in some cases were essential).

Material supplementing the present text may be found in these papers as follows: A list of nine examples of formal systems will be found in [OFP]; of these the ninth example, a formalization of ordinary polynomials, is revised in the Appendix (which dates from 1947). Another example, a formalization of a postulate set for group theory due to Dickson [DGF], is treated in essentially the same way in [APM] § 3, [TFD] I § 2, and [LLA] II § 2. More complex examples occur in [FRA] and [LSF]. Some examples of epitheorems (to be discussed in Chapter 2) are listed in [OFP] Chapter IX and [APM] § 6. For the definition of mathematics and criticism of intuitionistic and other idealistic views see [OFP] Chapters III and X, [APM] §§ 2 and 7. For the ontology and various representations of a formal system see [OFP] Chapter VI and [APM] § 4; the representability of a formal system in terms of a language with two symbols is shown in [LLA] Appendix § 3. The acceptability and relation to reality of a formal system are dealt with in [OFP] Chapter XI, [APM] § 8, and [TEx]; the relations to logic in [OFP] Chapter XII and [LLA] Introd.; the relations to semiotics (see also § 2) in [OFP] Chapter VIII, [APM] § 5, [MSL], [LFS], [LMF], [LSF], [LLA] Chapter I, and [SFL].

Some improvements in the conception of a formal system due to Rosser will be taken up in Chapter 7 below; while the relations to recursiveness will concern us in the second volume.

In his recent book [IMM] Kleene uses the term 'formal system' in a different sense from that employed here. Kleene's formal system is definitely syntactical. What he calls a generalized arithmetic ([IMM], § 50) is, however, a special kind of formal system in the present sense. The book contains some thoughtful and illuminating discussion of formal methods. We cite here, in particular, the following: the account of formalization on pp. 59–65; the notion of a deduction and its analysis pp. 86–89; generalized arithmetics already cited; and inductive definitions pp. 258–261. The book is also an important source of information regarding epitheoretical methods involving arithmetization.

The following items bear on the subject of this chapter, but were received too late to be taken into account: Lorenzen [EOL], Heyting [Int], Markov [TAlg – 1954]. The second edition of Church [IML] has been announced, but is not yet available to us.

2. Metasystems

The notion of a syntactical system was dismissed with a few remarks in § D5 because combinatory logic, which is our principal business, is presented as an abstract system. But, since by far the greater part of the current literature on formal methodology is written from a semiotical point of view, it will be appropriate to add here a few supplementary remarks. We refer to the literature for the details; but we shall discuss here certain questions in which it seems to us that the literature needs amplification, if not correction.

Suppose we have given a "language" $\mathfrak{L}$ in the generalized semiotical sense of § D1. A theory in which the expressions of $\mathfrak{L}$, or ways of constructing them (cf. the "composés" of [LLA] Chapter 1 § 5), are taken as obs is called a *metatheory* of $\mathfrak{L}$; if the theory is formalized, at least partially, it may be called a *metasystem*. The A-language of such a metasystem is what we call a *metalanguage* (over $\mathfrak{L}$). In the literature the term 'metalanguage' is also used in the sense of what we here call

56. The manuscript was examined in May, 1940 by Kleene.

"U-language"; and there is a certain amount of confusion between these two senses (cf. [LFS], [LMF]).

The consideration of metasystems is thus a part of the study of languages in general. This study was called *semiotics* by Morris [FTS] (cf. Carnap [ISm] p. 9). He divided it into three parts. The first part, called *syntactics*, deals with the expressions of the language as physical objects, and admits only considerations which have reference solely to the arrangement and kinds of symbols of which the expressions are composed. The second part, called *semantics*, takes meanings into account; the third part, called *pragmatics*, takes into account actual use.

We wish now to point out that the distinction between syntactics and semantics is by no means as sharp as would appear from this definition. Indeed, as these terms are actually used, they are not only somewhat vague, but they certainly overlap with one another. We shall base our discussion on Carnap [ISm], and shall presuppose acquaintance with that work.

We begin by noticing—without bothering too much about the precise meaning of 'meaning'—that the semantical element may enter into a language by stages, so to speak. Thus 'sentence' and 'true' are words of the original U-language having significance therein. If we know what expressions of $\mathfrak{L}$ are sentences we know something semantical concerning $\mathfrak{L}$; this is so a fortiori if we also know what sentences are true. From this standpoint the A-language of a formal system is semantical; on the other hand the categories of 'sentence' and 'true sentence' for such an A-language are purely syntactical. Again, we may have semantical information which is accidental in the same sense that a representation of a formal system is accidental. Thus in certain examples given by Carnap [ISm] as semantical the notion of L-truth is syntactically definable; and in the examples actually given ordinary truth is also.

Now let us examine more closely the meaning of 'semantical'. According to Carnap l.c. p. 24, a semantical system admits three kinds of rules, viz.: rules of formation, [57] rules of truth, and rules of designation. Suppose we break semantics into three parts according to which sorts of rules are present. If only rules of formation are present we speak of *grammatics*; if rules of truth, of *aletheutics*; if rules of designation, of *onomatics*. Evidently an aletheutical system will necessarily be grammatical; if designation is understood in the generalized sense of Carnap § 12, then Carnap shows that an onomatical system is aletheutical. Whether there is any point in breaking what is here called onomatics into two further stages (as was suggested in [LFS] and [LMF]) is uncertain; likewise the possibility of considering two sorts of designation relation, as in Frege's [SBd] (cf. Church [FLS]) is not gone into.

Now, where does semantics begin? It must be admitted that there is a major break between aletheutics and onomatics; for if a metatheory of the latter sort is formalized, there will be obs to be interpreted as expressions (or ways of constructing them), as well as obs whose interpretations are the designata of these expressions, which would not be the case for an aletheutical theory. On the other hand a large part of Parts B and C of Carnap's book is concerned with aletheutics; in particular §§ 9, 14, 18, and several other sections deal with theorems about an arbitrary aletheutical system. Indeed, this and several other factors give the impression that Carnap intended to include aletheutics under semantics. We see no reason why grammatics should not be included also.

We now examine the term 'syntactics'. This is used by Carnap in two senses. The broader of these two senses corresponds to the sense in which we have just used the word. This includes, then, such theories as those of Post [FRG] and Markov [TAl], as well as Church's theory of lambda-conversion in the way he himself expounds it in [CLC]. [58] The narrower sense Carnap introduces on pp. 155 ff. In this sense a syntactical system, or "calculus", is a system with rules of formation and rules of consequence. Since rules of consequence may be regarded as a form of rules of truth (cf. [LFM] p. 351), such a system is aletheutical. It is the syntactical counterpart of a logistic semiformal system.

57. The rules of formation tell what expressions are sentences.
58. It is expounded as a formal system in Chapter **3**.

All of this indicates that a redefinition of some basic semiotical concepts is desirable. Our suggestion is to use 'syntactics' and 'semantics' in the broad senses just outlined. Semantics then includes grammatics, aletheutics, and onomatics. Then a syntactical system may or may not be semantical—under appropriate circumstances it may belong under any of the dimensions of semantics (under onomatics when the properly onomatical information is accidental). The systems considered by Carnap in Part B (at least §§ 9, 14, 18, etc.) are aletheutical and syntactical (see [LSF])as well as those in Part D. We do not distinguish between these two forms of aletheutical systems, although the latter might be called deductive, the former schematic—because the rules are all axiom schemes. The same distinction can be made for formal systems, but is hardly fundamental. A system can be formulated in schematic form if and only if it is decidable. (Cf. Examples 3–5 in [OFP] Chapter V.)

We have seen in § D5 that a formal system and a syntactical system are essentially equivalent notions. For further details of this and a discussion of their relative advantages see the references at the end of § 1. As already stated, most of the current literature on formal methodology concerns syntactical systems which are only partially formalized; this can be transformed into literature about formal systems by completing the formalization and dropping the object language.

The following references concern those aspects of semiotics which are of interest from the standpoint of formal systems. For semiotics in general see Morris [FTS], [SLB]; Carnap [ISm], [FLg]. For syntactics especially see also Carnap [LSL], Lewis [SSL] Chapter VI, Scholz [WIK], Tarski [ILM], Quine [CBA], Rosenbloom [EML] pp. 152–193, 205–208 (describes the work of Post, which in turn shows an influence by Lewis). The "theory of algorithms", developed by Markov and his school, is treated briefly in Markov [TAl–1951] and at greater length in [TAl–1954]; the latter work did not reach us in time to be taken into account here. For the formalization of syntactics, see also Tarski [BBO]; Hermes [Smt]; Schröter [AKB], [WIM]. For methodological questions see also Tarski [FBM], [GZS]. The essential paper on Tarski's semantical definition of truth is his [WBF]; see also McKinsey [NDT]. The term 'grammatics' was introduced in [MSL]; for its substance see Ajdukiewicz [SKn]; also [FPF], [TFD] Chapter I, Bocheński [SCt], Bar-Hillel [SCt]. For the notions of U-language, A-language, and metalanguage see especially [LFS], [LMF], [TFD]; these papers and [MSL] are to be consulted for the development of the ideas sketched here on the subdivision of semiotics. Papers on the philosophy of meaning, arguments for and against nominalism, etc. are not included here.[58a]

3. Formal systems and nominalism

In the foregoing we have gone to some pains to show that the notion of formal system can be understood from various philosophical viewpoints. We consider here a certain extreme viewpoint, that of nominalism, for which the notion requires some modification.

58a. (Added in proof.) We defer to later publications amendments to this article due to occurrences during printing. However, the following points need to be mentioned: For more historically oriented references than those given here (cf. the Preface), see Church [IML_2]. The overlap between syntactics and semantics is recognized in Tarski [WBF]. The discussion in Church (l.c.) convinces us that one can hardly make any reasonable sharp distinction with respect to a nonformalized metatheory. With respect to a formal metasystem, as here defined, a distinction is conceivable; but the epitheory of a syntactical metasystem may still be semantical.

We have seen that a formal system can be interpreted in terms of the well-formed expressions of a certain object language. From the nominalistic standpoint even the notion of expression—in the sense of Carnap's "expression-design"[59]—is an abstract concept, as one may have different instances of the same expression. One replaces it by the notion of *inscription*, which is a single concrete instance of an expression. The different instances of an expression are then different inscriptions; but we can say that they are similar, or *equiform* (Leśniewski's "gleichgestaltet").

From a nominalist standpoint, then, we can regard an ob as a category of similar well-formed inscriptions. Let us call a particular occurrence of such an ob a *cob*. Then we can imagine a machine which will print a series of standard cobs, let us say upon a paper tape. When a formation rule allows the construction of an ob from previous obs, the machine uses for the latter purpose obs which are equiform to previously printed standard cobs. In this way an ob can be identified with a standard cob, and instances of that ob with equiform cobs.

All this requires some changes in the description of a formal system in the U-language. A number of other changes of similar character must also be made. But these changes are rather trivial, and can be made systematically. There is no trouble about describing a formal system in nominalistically acceptable terms.

More interesting is the question of when the object language of such a formal system can be nominalistically interpreted. This is particularly important for the theory of combinators, which involves primitive obs like K and S (see Chapters **5–7**) whose conceptual interpretations are notions of a high degree of abstraction. But the presence of abstract terms does not necessarily make a theory nominalistically unacceptable. It is plausible that, given any theory $\mathfrak{S}$ whatever, we can find a theory $\mathfrak{S}'$, based on the theory of combinators, which is equivalent (in a sense we shall not endeavor to make precise) to $\mathfrak{S}$. If we accept this as a heuristic principle, then, if there is a nominalistically acceptable theory at all, there will be an equivalent one based on the theory of combinators. The question of nominalistic interpretation of such a theory admittedly involves problems; but they do not seem to us to be insoluble.

The criterion, proposed by Quine [Unv] for testing the "ontological commitment" of a theory based on first-order predicate calculus, is the nature of the objects which can be values of the bound variables; free variables cause no difficulty. This criterion is not applicable to theories based on a synthetic theory of combinators; because in such a theory there are no variables of any kind. Generality, whether expressed ordinarily by bound variables or free, has to be expressed in combinatory logic by explicit quantifiers. In order to apply Quine's criterion, it is necessary to resort to a translation of some kind, and it is not clear exactly how such a translation should be made.

Although we think it is premature to attempt a definitive answer to this question, we suggest the following tentative solution. The universal quantifier, or something essentially similar, is used for the kind of generality which is ordinarily expressed by free variables; and we need this kind of quantifier in order to express in finite form the axioms and rules of the translated theory. But the rules for the introduction of the existential quantifier are not necessary for this fundamental stage; and they probably require restriction, in any event, if contradiction is to be avoided. It is therefore natural to make the parting of the ways occur at the point where an existential quantifier is introduced. In the original theories of Church, and in the set-theoretic system of Cogan [FTS], the existential quantifier Σ is introduced in such a way as to imply that sets or other abstract objects exist; and therefore these theories are not nominalistically acceptable. But we see no reason why a nominalistically acceptable theory could not be constructed.

This question does not come up in this volume because we have no existential

59. A similar term was used by Leśniewski (cf. Tarski [WBF] pp. 267–269 footnotes 3 and 5).

quantifier. Except for the present article we therefore pay no attention to the demands of nominalism. It is quite plausible that the conceptualistic manner of speaking which we employ can be translated into a nominalistic one. But the question of exactly how to make such a translation we leave open.

For nominalistic theories in general we refer to Goodman [SAp] and to various papers by Quine collected in his [LPV].[60]

60. For various suggestions in connection with the foregoing we are indebted to P. C. Gilmore; but he is not responsible for the views expressed.