1. Pattern matching with regular expressions
   it’s often useful to search strings for patterns
   (e.g. find all sentences containing two commas)

   Regular Expressions provide a nice shorthand for such patterns
   all regular expressions can be recognized by FSAs

2. Regular expression syntax
   a Regular Expression (RE) $\rho$ is a string made up of:
   
   observation symbols: $x$  
   e.g.: $a$  
   language: $\{ a \}$
   
   concatenations of REs: $\rho \rho''$  
   e.g.: $ab$  
   language: $\{ ab \}$
   
   disjunctions of REs: $(\rho' \mid \rho'')$  
   e.g.: $(ab \mid b)$  
   language: $\{ ab, b \}$
   
   ‘Kleene star’ repetitions of RE: $(\rho')^*$  
   e.g.: $(ab)^*$  
   language: $\{ \epsilon, ab, abab, ababab, \ldots \}$

   (epsilon $\epsilon$ means the empty string)

   For example:

   $\text{the } (\text{dog} \mid \text{cat} \mid \text{rat}) \ (\text{that} \ (\text{chased} \mid \text{ate} \mid \text{nibbled}) \ \text{the} \ (\text{cat} \mid \text{rat} \mid \text{malt}))^*$

   recognizes the following sentences:

   \{the rat,
   the rat that nibbled the malt,
   the cat that ate the rat that nibbled the malt,
   the dog that chased the cat that ate the rat that nibbled the malt that ate the rat that chased the cat,
   \ldots \}

   REs are often augmented with the following (equiv. to combinations of concat, disjn, star):

   wildcard symbols: $. = (x \mid x' \mid x'' \ldots)$  
   e.g.: $. $  
   lang: $\{ a, b, c, d, \ldots \}$
   
   symbol disjunctions: $[xx'x''] = (x \mid x' \mid x'')$  
   e.g.: $[ace]$  
   lang: $\{ a, c, e \}$
   
   symbol ranges: $[x - x'] = (x \ldots x')$  
   e.g.: $[a-e]$  
   lang: $\{ a, b, c, d, e \}$
   
   one or more repetitions of REs $(\rho')^+ = \rho'(\rho')^*$  
   e.g.: $(ab)^+$  
   lang: $\{ \text{ab, abab, ababab, } \ldots \}$
   
   zero or one repetitions of REs $(\rho')? = (\rho'|\epsilon)$  
   e.g.: $(ab)?$  
   lang: $\{ \epsilon, ab \}$

   Most RE implementations assume $. * p . * \text{, let anchors } ^\wedge \text{ and } ^\$ \text{ match beginning/end of line}$

3. We can build an FSA $FSA(\rho)$ that accepts the same strings as any RE $\rho$
   in other words, $\forall \rho . L(FSA(\rho)) = L(\rho)$
   in other words, $L(RE) \subseteq L(FSA)$
   base case – for observations in RE:
• observation symbols $x$:

$$FSA(x) = \langle \{q, q'\}, \{x\}, \{q\}, \{q', \langle q, x, q' \rangle\} \rangle$$

graphically:

![Graphical representation of observation symbols](image)

inductive step – combine REs using ‘$\epsilon$-transitions’ w/o associated obs, then compile out (assume state sets of sub-expressions $Q_{FSA(\rho_1)}$ and $Q_{FSA(\rho_2)}$ are disjoint):

• concatenations of REs $\rho_1, \rho_2$:

$$FSA(\rho_1 \rho_2) = \langle Q_{FSA(\rho_1)} \cup Q_{FSA(\rho_2)},$$

$$X_{FSA(\rho_1)} \cup X_{FSA(\rho_2)},$$

$$S_{FSA(\rho_1)},$$

$$F_{FSA(\rho_2)},$$

$$M_{FSA(\rho_1)} \cup M_{FSA(\rho_2)} \cup \{\langle q', \epsilon, q''\rangle \mid q' \in F_{FSA(\rho_1)}, q'' \in S_{FSA(\rho_2)}\} \rangle$$

graphically:

![Graphical representation of concatenation](image)

• disjunctions of REs $\rho_1, \rho_2$:

$$FSA(\rho_1 \mid \rho_2) = \langle Q_{FSA(\rho_1)} \cup Q_{FSA(\rho_2)},$$

$$X_{FSA(\rho_1)} \cup X_{FSA(\rho_2)},$$

$$S_{FSA(\rho_1)} \cup S_{FSA(\rho_2)},$$

$$F_{FSA(\rho_1)} \cup F_{FSA(\rho_2)},$$

$$M_{FSA(\rho_1)} \cup M_{FSA(\rho_2)} \rangle$$
• Kleene star repetitions of RE $\rho$:

$$FSA(\rho^*) = \langle Q_{FSA(\rho)} \cup \{q, q''\}, X_{FSA(\rho)}, \{q\}, \{q''\}, M_{FSA(\rho)} \cup \{(q, \epsilon, q') \mid q' \in S_{FSA(\rho)}\} \cup \{(q'', \epsilon, q'') \mid q'' \in F_{FSA(\rho)}\} \cup \{(q, \epsilon, q''), (q'', \epsilon, q)\} \rangle$$

graphically:
for example:

\[
\begin{align*}
\text{dog} & \quad \epsilon \\
q_1 & \quad \rightarrow \quad q_2 \\
q_2 & \quad \epsilon \\
q_3 & \quad \epsilon \\
q_4 & \quad \epsilon \\
q_5 & \quad \epsilon \\
q_6 & \quad \epsilon \\
q_7 & \quad \epsilon \\
q_8 & \quad \epsilon \\
q_9 & \quad \epsilon \\
q_{10} & \quad \epsilon \\
q_{11} & \quad \epsilon \\
q_{12} & \quad \epsilon \\
q_{13} & \quad \epsilon \\
q_{14} & \quad \\
\end{align*}
\]

finally, remove \(\epsilon\)-transitions — this is an algorithm, a procedure for computing something:

(a) \(\epsilon\) closure – add shortcuts for progressively longer chains of \(\epsilon\)-transitions:

\[
M_A^0 = M_A
\]

for each chain length \(k\) from 1 to \(|Q|\):

\[
M_A^k = M_A^{k-1} \cup \{ \langle q, \epsilon, q'' \rangle \mid \langle q, \epsilon, q' \rangle \in M_A^{k-1}, \langle q', \epsilon, q'' \rangle \in M_A \}
\]

(b) merge \(\epsilon\)-transitions with labeled transitions, start/final states to get new automaton \(A'\):

\[
A' = \langle Q_A, \\
X_A, \\
S_A \cup \{ q' \mid \exists q. q \in S_A, \langle q, \epsilon, q' \rangle \in M_A^{\mid Q_A} \}, \\
F_A \cup \{ q \mid \exists q'. \langle q, \epsilon, q' \rangle \in M_A^{\mid Q_A}, q' \in F_A \}, \\
\{ \langle q, x, q' \rangle \mid \langle q, x, q' \rangle \in M_A, x \in X_A \} \cup \{ \langle q, x, q'' \rangle \mid \langle q, \epsilon, q' \rangle \in M_A^{\mid Q_A}, \langle q', x, q'' \rangle \in M_A \}
\]

for example (ignoring unconnected states):
4. Practice:
Write a regular expression to recognize the infinite language containing the following (treat each word as a single symbol):

- hello ok bye
- hello ok ok bye
- hello ok ok ok bye
- hello ok ok ok ok bye

5. FSAs also closed under the following operations (so REs could support them):

- **reversal of RE** $\rho$: (change direction of all arrows)
  
  $$FSA(\rho^R) = \langle Q_{FSA(\rho)}, X_{FSA(\rho)}, F_{FSA(\rho)}, S_{FSA(\rho)}, \{(q', x, q) | (q, x, q') \in M_{FSA(\rho)}\} \rangle$$

- **negation of RE** $\rho$: (swap final and non-final states)
  
  $$FSA(\neg \rho) = \langle Q_{FSA(\rho)}, X_{FSA(\rho)}, S_{FSA(\rho)}, Q_{FSA(\rho)} - F_{FSA(\rho)}, M_{FSA(\rho)} \rangle$$

- **conjunction of REs** $\rho_1, \rho_2$: (use pairs of sub-expression states)
  
  $$FSA(\rho_1 \land \rho_2) = \langle Q_{FSA(\rho_1)} \times Q_{FSA(\rho_2)}, X_{FSA(\rho_1)} \cap X_{FSA(\rho_2)}, \{(q, q') | q \in S_{FSA(\rho_1)}, q' \in S_{FSA(\rho_2)}\}, \{(q, q') | q \in F_{FSA(\rho_1)}, q' \in F_{FSA(\rho_2)}\}, \{\{(q, q''), x, \langle q', q'''\rangle) | (q, x, q') \in M_{FSA(\rho_1)}, (q'', x, q''') \in M_{FSA(\rho_2)}\} \rangle$$

- **exclusion of REs** $\rho_1, \rho_2$: (combine negation and conjunction)
  
  $$FSA(\rho_1 - \rho_2) = FSA(\rho_1 \land \neg \rho_2)$$

6. Limits of FSAs / REs:
FSAs (and therefore REs) can only recognize sequences with finitely-bounded memory

**Pumping lemma:**
if $L$ is an infinite regular language (in $L(FSA)$), then $\exists x, y, z$ such that $y \neq \epsilon$ and $xy^n z \in L$ for all $n \geq 0$

(where $y^n$ means $n$ repetitions of string $y$)
Exception: \(a^n b^n\): \(\{\epsilon, ab, aabb, aaabbb, \ldots\}\) is not regular

why not?
because, in order to allow infinite languages with finites states, \(y\) must occur either...

- within the a’s, generating strings like \(aaaaabbb\) when pumped, or
- within the b’s, generating strings like \(aaabbbb\) when pumped, or
- within the crossover from a’s to b’s, generating strings like \(aaababbb\) when pumped

none of which are in \(a^n b^n\)

NOTE: the same problem comes up in trying to recognize nested parentheses!

7. Cognitive plausibility of FSAs

- problem for FSAs – we seem to learn general syntactic patterns w. unbounded nesting:
  ‘[NP [NP the photos ] [NP the reporter ] [v took ] ] were good’ \((NP \rightarrow NP \ NP \ V)\)
  when center NP is expanded, this generates non-regular language \(NP^n \ NP \ V^n\):
  e.g. ‘[NP the photos ] [NP [NP the reporter ] [NP I ] [v hired ] ] [v took ] were good’
- but in practice – we can’t keep track of more than 4 or so disconnected ideas:
  ‘the malt the rat the cat the dog the man I know bought bit chased ate was rancid’

this is called a ‘competence / performance’ distinction: we are FSAs emulating non-FSAs