Ling 5801: Lecture Notes 8
From CFGs to Pushdown Automata

We can recognize CFGs using ***pushdown automata***.

Contents

8.1 PDAs are FSAs extended with an infinite pushdown store at each state

A Pushdown Automaton (PDA) is a tuple $\langle Q, X, S, F, M \rangle$, where:

- Q is a finite set of states
- X is a finite set of observation symbols
- $S \subseteq Q$ is a set of start states
- $F \subseteq Q$ is a set of final states
- $M \subseteq Q \times (Q \cup \{\epsilon\}) \times (X \cup \{\epsilon\}) \times Q \times (Q \cup \{\epsilon\})$ is a set of store,state transitions, of form:
  - $\langle q, \epsilon, \epsilon, q', q \rangle$ — called an 'expansion' (or 'stack push'):

    transition from state q to q', replacing empty string ϵ at front of store with q
  - $\langle q, q'', x, q', q'' \rangle$ — called a 'state transition':

    transition from state q to q' on observation x, replacing q'' at front of store with q''
  - $\langle q, q'', \epsilon, q', \epsilon \rangle$ — called a 'reduction' (or 'stack pop/pull'):

    transition from state $q \in F$ to q', replacing q'' at front of store with empty string ϵ

From 'Candyland' to 'Trivial Pursuit': pieces carry stacked-on sub-pieces corresp. to spaces

8.2 Language accepted by PDA $A = \langle Q_A, X_A, S_A, F_A, M_A \rangle$

The automaton nondeterminisitically explores all possible states combined with all possible stores:

- The automaton starts at a start state with an empty store.

- The automaton pushes and pulls only to/from the front of the store.
- The automaton accepts the input on an empty store at any final state.

$L(A) = \{x_{1..T} \mid q \in S_A,\ \langle q, \epsilon\rangle \in V_A(x_{1..T})\}$

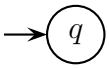
where $V_A(x_{t..T})$ returns a set of state,store pairs from which observations $x_{1..T}$ are acceptable:

$$\begin{aligned} V_A(x_{t..T}) = &\{\langle q, \epsilon\rangle \mid q \in F_A,\ x_{t..T} = \epsilon\} \\ &\cup \{\langle q, q''\alpha\rangle \mid \langle q, q'', x_t, q', q''\rangle \in M_A,\ \langle q', q''\alpha\rangle \in V_A(x_{t+1..T})\} \\ &\cup \{\langle q, \alpha\rangle \mid \langle q, \epsilon, \epsilon, q', q\rangle \in M_A,\ \langle q', q\alpha\rangle \in V_A(x_{t..T})\} \\ &\cup \{\langle q, q''\alpha\rangle \mid \langle q, q'', \epsilon, q', \epsilon\rangle \in M_A,\ \langle q', \alpha\rangle \in V_A(x_{t..T})\} \end{aligned}$$

8.3 Graphical representation of PDAs

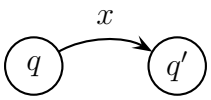
PDAs can be represented graphically:

- start states: $q \in S$



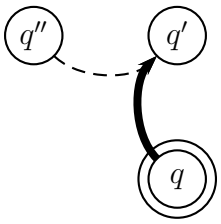
- final states: $q \in F$

- state transitions: $\langle q, \epsilon, x, q', \epsilon\rangle \in M$

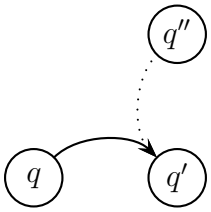


- expansions: $\langle q, \epsilon, \epsilon, q', q\rangle \in M$ (like transitions, but push previous state onto store)

- reductions: $\langle q, q'', \epsilon, q', \epsilon\rangle \in M, \quad q \in F$ (like trans, but remove last state from store)



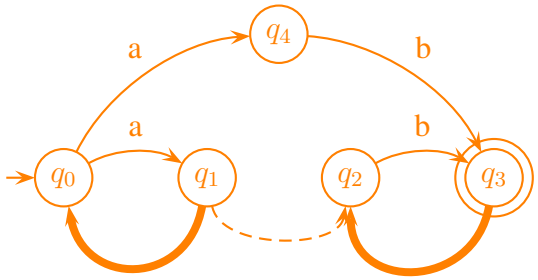
- conditional transitions: $\langle q, q'', \epsilon, q', q'' \rangle \in M$



8.4 Example PDA for: $a^n b^n; n > 0$ (S $\rightarrow$ a S b, S $\rightarrow$ a b)

Q:$\{q_0, q_1, q_2, q_3, q_4\}$, X:{a,b}, S:$\{q_0\}$, F:$\{q_3\}$,

M:$\{\langle q_0, \epsilon$,a, $q_1, \epsilon\rangle, \langle q_2, \epsilon$,b, $q_3, \epsilon\rangle, \langle q_0, \epsilon$,a, $q_4, \epsilon\rangle, \langle q_4, \epsilon$,b, $q_3, \epsilon\rangle, \langle q_1, \epsilon, \epsilon, q_0, q_1\rangle, \langle q_3, q_1, \epsilon, q_2, \epsilon\rangle\}$



8.5 PDAs can recognize CFGs; $\mathcal{L}$(CFG) $\subseteq \mathcal{L}$(PDA)

Given any CFG $\langle C_G, X_G, S_G, R_G \rangle$, we can define a PDA $\langle Q, X, S, F, M \rangle$

(assume binary-branching CFG, since all CFGs can be translated into one):

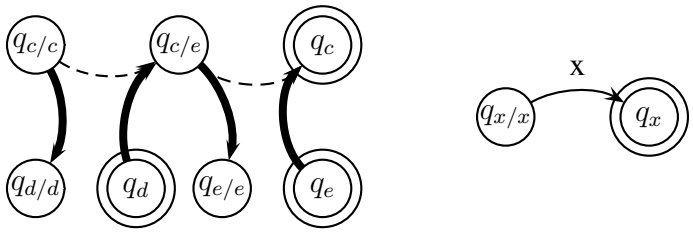
$Q = \{q_{c/c}, q_{c/e}, q_c \mid c \rightarrow d\ e \in R_G\} \cup \{q_{x/x}, q_x \mid x \in X_G\}$

$X = X_G$

$S = \{q_{c/c} \mid c \in S_G\}$

$F = \{q_c \mid c \to d\ e \in R_G \text{ or } c \in X_G\}$

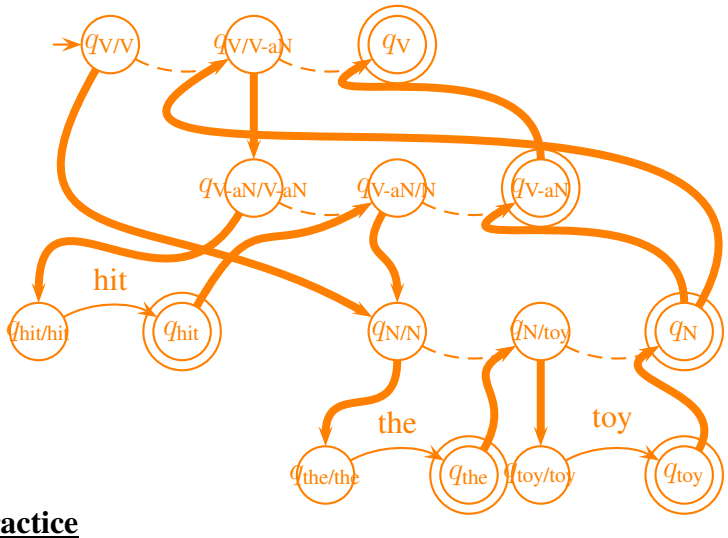
$M = \{\langle q_{c/c}, \epsilon, \epsilon, q_{d/d}, q_{c/c}\rangle,$
$\langle q_d, q_{c/c}, \epsilon, q_{c/e}, \epsilon\rangle,$
$\langle q_{c/e}, \epsilon, \epsilon, q_{e/e}, q_{c/e}\rangle,$
$\langle q_e, q_{c/e}, \epsilon, q_c, \epsilon\rangle \mid c \to d\ e \in R_G\} \cup \{\langle q_{x/x}, \epsilon, x, q_x, \epsilon\rangle \mid x \in X_G\}$



for example, a grammar with the following rules:

V $\to$ N V-aN
V-aN $\to$ hit N
N $\to$ the toy

would yield the following PDA:



Practice

How would the store look after each expansion, reduction, or transition in the above PDA?

$q_{V/V}$	(start state)
$q_{N/N}$ $q_{V/V}$	(expand, pushing $q_{V/V}$ onto store)
. . . ?	

8.6 Right-expansion elimination and left-expansion elimination

Easy to implement recognizer if only one expansion, reduction per transition

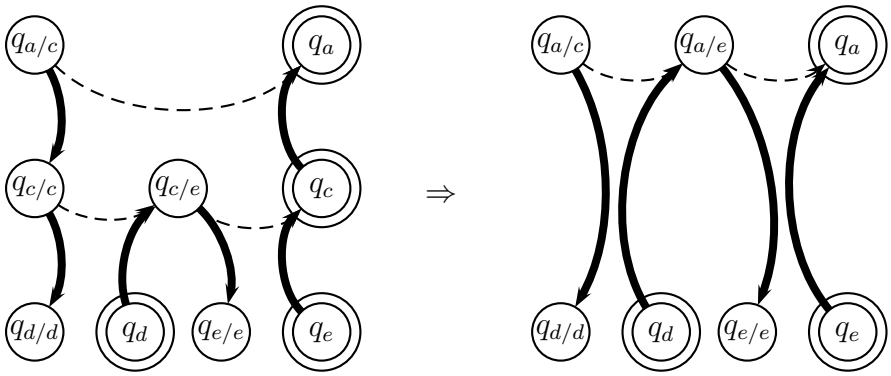
1. Right-expansion elimination ('awaited transition') – right child 'c' disappears:

   $M^{(0)} = M$

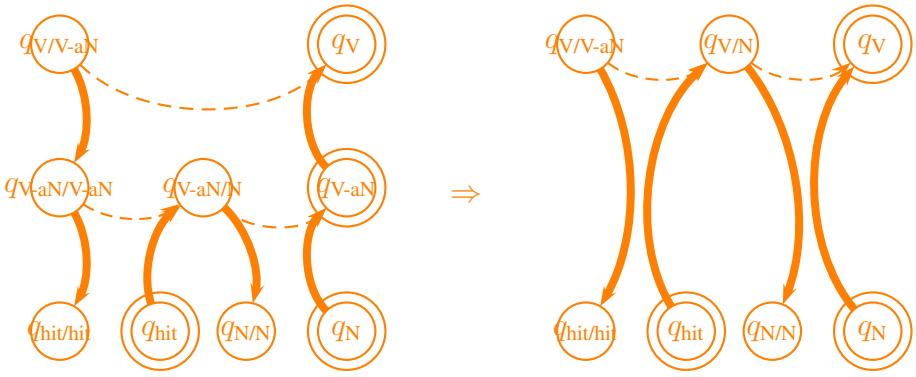
   $M^{(k)} = M^{(k-1)} \cup \{ \langle q_{a/c}, \epsilon, \epsilon, q_{d/d}, q_{a/c} \rangle, \langle q_d, q_{a/c}, \epsilon, q_{a/e}, \epsilon \rangle,$
   $\langle q_{a/e}, \epsilon, \epsilon, q_{e/e}, q_{a/e} \rangle, \langle q_e, q_{a/e}, \epsilon, q_a, \epsilon \rangle$
   $| \langle q_{a/c}, \epsilon, \epsilon, q_{c/c}, q_{a/c} \rangle \in M^{(k-1)}, \langle q_{c/c}, \epsilon, \epsilon, q_{d/d}, q_{c/c} \rangle \in M^{(k-1)},$
   $\langle q_d, q_{c/c}, \epsilon, q_{a/e}, \epsilon \rangle \in M^{(k-1)}, \langle q_{c/e}, \epsilon, \epsilon, q_{e/e}, q_{c/e} \rangle \in M^{(k-1)},$
   $\langle q_e, q_{c/e}, \epsilon, q_c, \epsilon \rangle \in M^{(k-1)}, \langle q_c, q_{a/c}, \epsilon, q_a, \epsilon \rangle \in M^{(k-1)} \}$

   $M' = M^{|R|}$

   Recognition is equivalent because child sub-models are preserved in order:



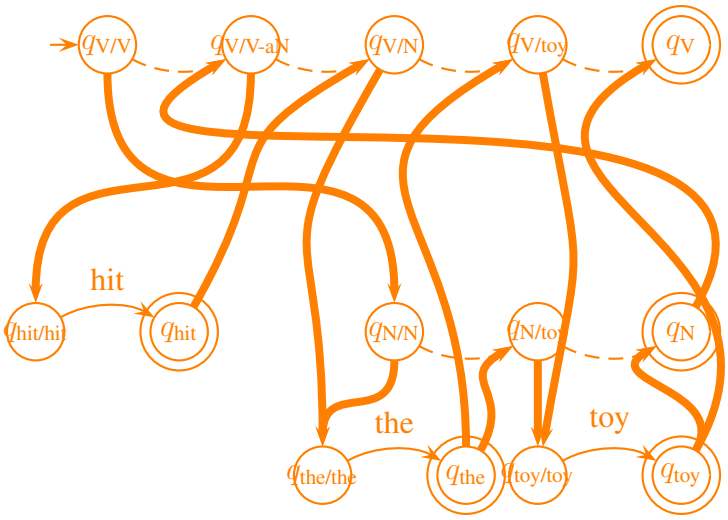
   for example:



   Repeat by mapping $q_{a/e}$, q_a in result to $q_{a/c}$, q_a in subsequent iteration.

   Now all sequences of reductions are replaced with a single reduction!

   Result (hiding unnecessary structure):



2. Left-expansion elimination ('active transition') – left child 'c' disappears:

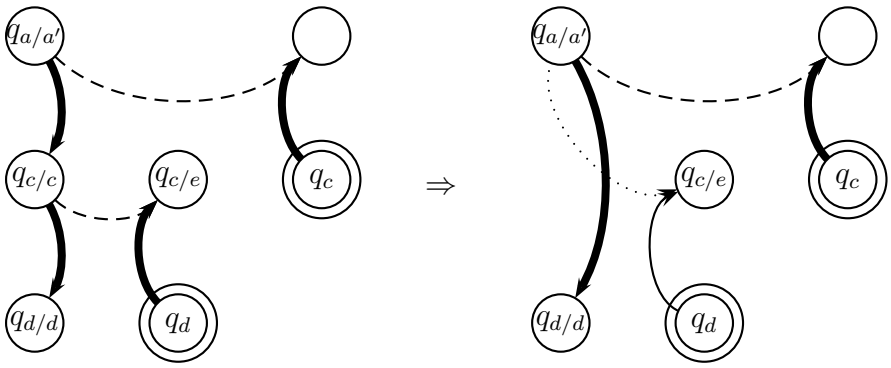
$$M^{(0)} = M$$

$$M^{(k)} = M^{(k-1)} \cup \{ \langle q_{a/a'}, \epsilon, \epsilon, q_{d/d}, q_{a/a'} \rangle, \langle q_d, q_{a/a'}, \epsilon, q_{c/e}, q_{a/a'} \rangle \mid \langle q_{a/a'}, \epsilon, \epsilon, q_{c/c}, q_{a/a'} \rangle \in M^{(k-1)}, \langle q_{c/c}, \epsilon, \epsilon, q_{d/d}, q_{c/c} \rangle \in M^{(k-1)}, \langle q_d, q_{c/c}, \epsilon, q_{c/e}, \epsilon \rangle \in M^{(k-1)} \}$$

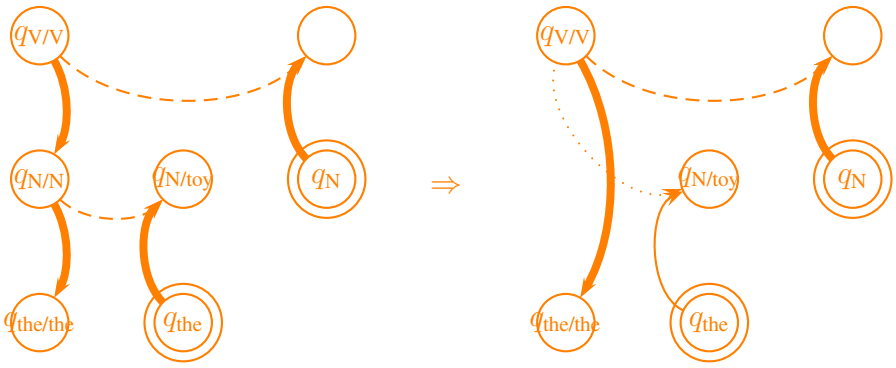
$$M' = M^{|R|}$$

Add 'conditional' ϵ-transition: $\langle q_d, q_{a/a'}, \epsilon, q_{c/e}, q_{a/a'} \rangle$ (drawn with dotted line)

Recognition is equivalent because child sub-models preserved in order:



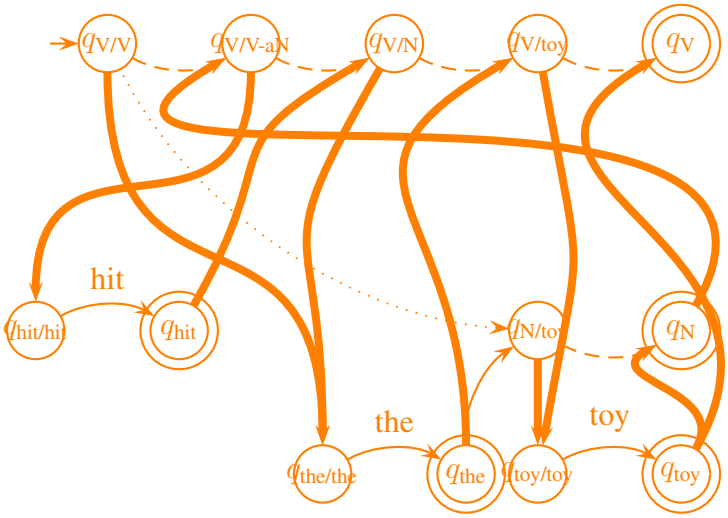
for example:



Repeat by mapping q_d, $q_{a/a'}$ in result to q_c, $q_{a/a'}$ in subsequent iteration.

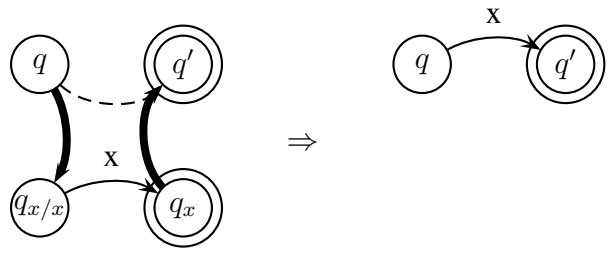
Now all sequences of expansions are replaced with a single expansion!

Result (hiding unnecessary structure):

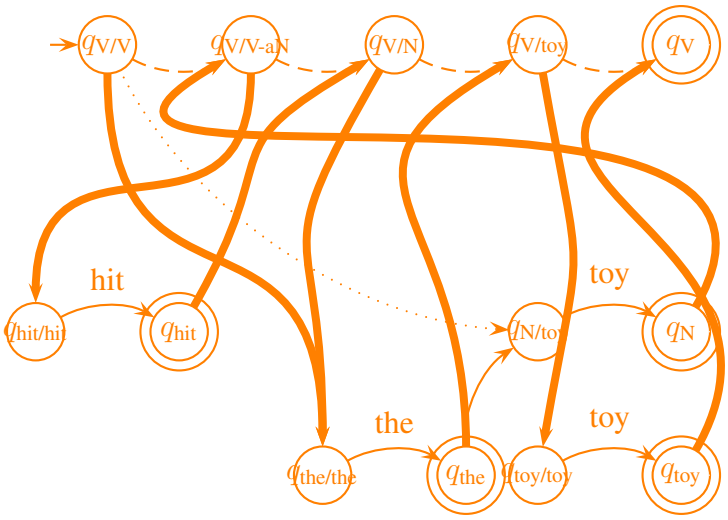


3. Transition flattening:

$M' = M \cup \{\langle q, \epsilon, x, q', \epsilon\rangle \mid \langle q, \epsilon, \epsilon, q_{x/x}, q\rangle, \langle q_{x/x}, \epsilon, x, q_x, \epsilon\rangle, \langle q_x, q, \epsilon, q', \epsilon\rangle \in M, q' \in F\}$

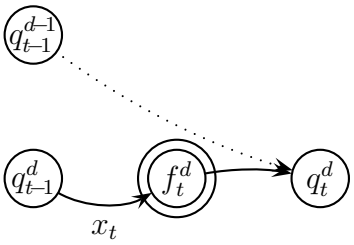


Result (hiding unnecessary structure):

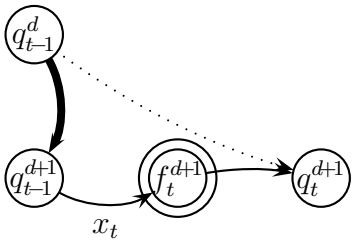


8.7 Implementation of PDA

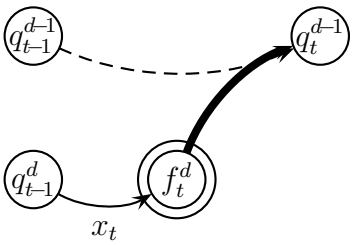
Since only one expansion/reduction between transitions, we only need four combinations:

a) 

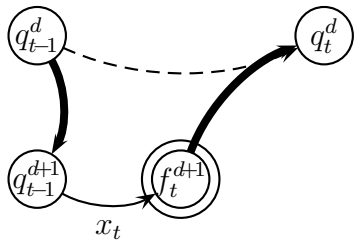
(no expand + trans + no reduce)

b) 

(expand + trans + no reduce)

c) 

(no expand + trans + reduce)

d)  (expand + trans + reduce)

Recognition algorithm:

for each time step t:
  for each previous state q^d_{t-1} and store σ (where $d-1$ is the depth of the store):
    a) for each final state f^d_t and current state q^d_t: (expand +) trans
    $V[t, q^d_t\sigma] = V[t, q^d_t\sigma]$ or $(V[t-1, q^d_{t-1}\sigma]$ and $M[q^d_{t-1}, \epsilon, x_t, f^d_t, \epsilon]$
    and $M[f^d_t, q^{d-1}_{t-1}, \epsilon, q^d_t, q^{d-1}_{t-1}])$
    b) for each previous state at deeper level q^{d+1}_{t-1}, final state f^{d+1}_t and current state q^{d+1}_t:
    $V[t, q^{d+1}_t q^d_t\sigma] = V[t, q^{d+1}_t q^d_t\sigma]$ or $(V[t-1, q^d_{t-1}\sigma]$ and $M[q^d_{t-1}, \epsilon, \epsilon, q^{d+1}_{t-1}, q^d_{t-1}]$
    and $M[q^{d+1}_{t-1}, \epsilon, x_t, f^{d+1}_t, \epsilon]$
    and $M[f^{d+1}_t, q^d_{t-1}, \epsilon, q^{d+1}_t, q^d_{t-1}])$
    c) for each final state f^d_t and current state q^{d-1}_t: (expand +) trans + reduce
    $V[t, q^{d-1}_t\sigma] = V[t, q^{d-1}_t\sigma]$ or $(V[t-1, q^d_{t-1}q^{d-1}_{t-1}\sigma]$ and $M[q^d_{t-1}, \epsilon, x_t, f^d_t, \epsilon]$
    and $M[f^d_t, q^{d-1}_{t-1}, \epsilon, q^{d-1}_t, \epsilon])$
    d) for each previous state at deeper level q^{d+1}_{t-1} and final state f^{d+1}_t and current state q^d_t:
    $V[t, q^d_t q^{d-1}_t\sigma] = V[t, q^d_t q^{d-1}_t\sigma]$ or $(V[t-1, q^d_{t-1}q^{d-1}_{t-1}\sigma]$ and $M[q^d_{t-1}, \epsilon, \epsilon, q^{d+1}_{t-1}, q^d_{t-1}]$
    and $M[q^{d+1}_{t-1}, \epsilon, x_t, f^{d+1}_t, \epsilon]$
    and $M[f^{d+1}_t, q^d_{t-1}, \epsilon, q^d_t, \epsilon])$

Correctness can be shown from definition of accepted languages.

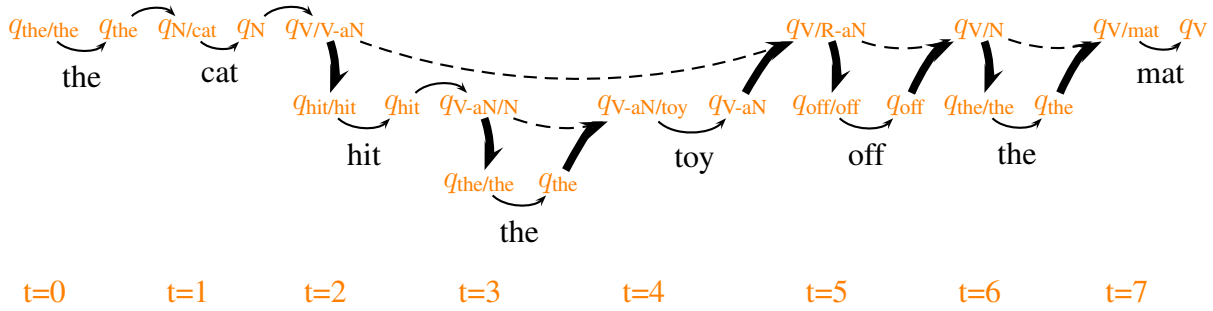
But, iterating over all possible infinite stores σ results in exponential complexity.

8.8 Example

Transforming the following CFG:

T → V T (a top-level discourse type)
V → N V-aN
V-aN → V-aN R-aN
V-aN → hit N
R-aN → off N
A-aN → off N
N → N A-aN
N → the cat
N → the toy
N → the mat

then recognizing 'the cat hit the toy off the mat' results in the following sequence:


$q_{\text{the/the}}$ q_{the} $q_{\text{N/cat}}$ q_{N} $q_{\text{V/V-aN}}$ $q_{\text{V/R-aN}}$ $q_{\text{V/N}}$ $q_{\text{V/mat}}$ q_{V}
the
cat
$q_{\text{hit/hit}}$ q_{hit} $q_{\text{V-aN/N}}$ $q_{\text{V-aN/toy}}$ $q_{\text{V-aN}}$ $q_{\text{off/off}}$ q_{off} $q_{\text{the/the}}$ q_{the}
hit
toy
off
the
$q_{\text{the/the}}$ q_{the}
the
t=0
t=1
t=2
t=3
t=4
t=5
t=6
t=7