Ling 5701: Lecture Notes 2
Linear Algebra

2.1 Terms

We can define matrices and vectors as arrays of real numbers:

- s is a **scalar** iff $s \in \mathbb{R}$

  You will often see scalars written as Greek letters, e.g.: γ

- $\mathbf{v}$ is a **vector** iff $\mathbf{v} \in \mathbb{R}^I$

  Scalars in vectors can be identified by one index: say $\mathbf{v} = \begin{bmatrix} 1.8 \\ -3 \end{bmatrix}$ then: $\mathbf{v}_{[2]} = -3$

- $\mathbf{M}$ is a **matrix** iff $\mathbb{M} \in \mathbb{R}^{I \times J}$

  Scalars in matrices can be identified by two indices: say $\mathbf{M} = \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix}$ then: $\mathbf{M}_{[2,1]} = -3$

2.2 Operations

- **transpose**: for all $\mathbf{M} \in \mathbb{R}^{I \times J}$, and all i, j indices to matrix rows and columns,

  $(\mathbf{M}^\top)_{[i,j]} = \mathbf{M}_{[j,i]}$

  For example: $\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix}^\top = \begin{bmatrix} 1.8 & -3 \\ 12 & 40 \end{bmatrix}$

- **scalar sum**: for all $s \in \mathbb{R},\ \mathbf{M} \in \mathbb{R}^{I \times J}$, and all i, j indices to matrix rows and columns,

  $(s + \mathbf{M})_{[i,j]} = (\mathbf{M} + s)_{[i,j]} = s + \mathbf{M}_{[i,j]}$

  (commutative)

  For example: $2 + \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} = \begin{bmatrix} 3.8 & 14 \\ -1 & 42 \end{bmatrix}$

- **matrix/vector sum**: for all $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{I \times J}$, with row and column indices i, j,

  $(\mathbf{M} + \mathbf{N})_{[i,j]} = (\mathbf{N} + \mathbf{M})_{[i,j]} = \mathbf{M}_{[i,j]} + \mathbf{N}_{[i,j]}$

  (commutative)

  For example: $\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2.8 & 14 \\ 0 & 44 \end{bmatrix}$

- **scalar product**: for all $s \in \mathbb{R},\ \mathbf{M} \in \mathbb{R}^{I \times J}$, with row and column indices i, j,

  $(s\,\mathbf{M})_{[i,j]} = (\mathbf{M}\,s)_{[i,j]} = s \cdot \mathbf{M}_{[i,j]}$

  (commutative)

  For example: $2 \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} = \begin{bmatrix} 3.6 & 24 \\ -6 & 80 \end{bmatrix}$

- **matrix/vector product**: for all $\mathbf{M} \in \mathbb{R}^{I \times K}, \mathbf{N} \in \mathbb{R}^{K \times J}$, with indices i, j, k,

  $(\mathbf{M}\,\mathbf{N})_{[i,j]} = \sum_k \mathbf{M}_{[i,k]} \cdot \mathbf{N}_{[k,j]}$

  (*not* commutative)

  For example:

$$\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \\ 15 & -6 \\ 7 & 18 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} (1.8 \cdot 1) + (12 \cdot 4) & (1.8 \cdot 2) + (12 \cdot 5) & (1.8 \cdot 3) + (12 \cdot 6) \\ (-3 \cdot 1) + (40 \cdot 4) & (-3 \cdot 2) + (40 \cdot 5) & (-3 \cdot 3) + (40 \cdot 6) \\ (15 \cdot 1) + (-6 \cdot 4) & (15 \cdot 2) + (-6 \cdot 5) & (15 \cdot 3) + (-6 \cdot 6) \\ (7 \cdot 1) + (18 \cdot 4) & (7 \cdot 2) + (18 \cdot 5) & (7 \cdot 3) + (18 \cdot 6) \end{bmatrix}$$

$$= \begin{bmatrix} 49.8 & 63.6 & 77.4 \\ 157 & 194 & 231 \\ -9 & 0 & 9 \\ 79 & 104 & 129 \end{bmatrix}$$

  There are two special cases of matrix multiplication for vectors:

  1. **inner ('dot') product**: for vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^I$,

     $\mathbf{v}^\top \mathbf{u} = \sum_i \mathbf{v}_{[i]} \cdot \mathbf{u}_{[i]}$

     For example:

$$\begin{bmatrix} 1.8 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1.8 \cdot 1) + (-3 \cdot 2)$$

$$= -4.2$$

  2. **outer product**: for vectors $\mathbf{v} \in \mathbb{R}^I, \mathbf{u} \in \mathbb{R}^J$,

     $(\mathbf{v}\,\mathbf{u}^\top)_{[i,j]} = \mathbf{v}_{[i]} \cdot \mathbf{u}_{[j]}$

     For example:

$$\begin{bmatrix} 1.8 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1.8 \cdot 1 & 1.8 \cdot 2 & 1.8 \cdot 3 \\ -3 \cdot 1 & -3 \cdot 2 & -3 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8 & 3.6 & 5.4 \\ -3 & -6 & -9 \end{bmatrix}$$

- **pointwise or Hadamard product**: for all $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{I \times J}$, with indices i, j,

  $(\mathbf{M} \odot \mathbf{N})_{[i,j]} = (\mathbf{N} \odot \mathbf{M})_{[i,j]} = \mathbf{M}_{[i,j]} \cdot \mathbf{N}_{[i,j]}$

  (commutative)

  For example:

$$\begin{bmatrix} 1.8 & 12 & 8 \\ -3 & 40 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1.8 \cdot 1 & 12 \cdot 2 & 8 \cdot 3 \\ -3 \cdot 4 & 40 \cdot 5 & 1 \cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8 & 24 & 24 \\ -12 & 200 & 6 \end{bmatrix}$$