LING4400: Lecture Notes 9
Set Theory vs. Type Theory

Contents

You may be familiar with notation and concepts from set theory [Cantor, 1874, Zermelo, 1908]:

- element: $x \in s$
- subset: $r \subset s$
- superset: $r \supset s$
- intersection: $r \cap s$
- union: $r \cup s$
- ...

People have used it to define foundational concepts in mathematics (over sets of numbers).

Type theory is like set theory in that it can serve as a basis for math, but it's more constrained.

9.1 Russell's paradox [Russell, 1902]

Why do we want a more constrained basis for logical meaning?

In basic set theory, a set can include *a-n-y-t-h-i-n-g*: entities, sets of entities, sets of sets...!

This power corrupts it!

For example, you can do this:

1. Define the set of all sets that do not contain themselves.
2. Now it contains itself only if it *does not contain itself.*

This is called **Russell's paradox**.

Type theory is instead defined by **construction**: complex types are defined in terms of simpler ones.

This means no type can contain itself.

(You can't even *ask* it in lambda calculus: An $\langle \mathrm{e}, \mathrm{t} \rangle$ can't apply to an $\langle \mathrm{e}, \mathrm{t} \rangle$ — it's ungrammatical!)

Good, because we'll never need or want that! In nature, things can't generally contain themselves.

9.2 Set theoretic functions in type theory

We can rescue some notation from set theory (un-Googlable symbols make us look smart!).

But we have to / get to assume the elements in our sets are all of the same type.

So we can notate sets of type $\langle \alpha, \mathrm{t} \rangle$ (with elements of type α) as:

1. **set roster** — type $\langle \alpha, \mathrm{t} \rangle$

$$[\![\{\varphi, \chi, \psi, \ldots\}]\!]^M = [\![\lambda_{x:\alpha}\ x = \varphi \lor x = \chi \lor x = \psi \lor \ldots]\!]^M$$

   for example, an expression meaning *the set containing Laos, Mali and Togo*:

$$[\![\{\mathsf{Laos}, \mathsf{Mali}, \mathsf{Togo}\}]\!]^M = [\![\lambda_{x:\mathrm{e}}\ x = \mathsf{Laos} \lor x = \mathsf{Mali} \lor x = \mathsf{Togo}]\!]^M =$$

input		output
Africa	:	**False**
Asia	:	**False**
Laos	:	**True**
Mali	:	**True**
Togo	:	**True**

2. **set comprehension** — type $\langle \alpha, \mathrm{t} \rangle$

$$[\![\{x{:}\alpha \mid \varphi\}]\!]^M = [\![\lambda_{x:\alpha}\ \varphi]\!]^M$$

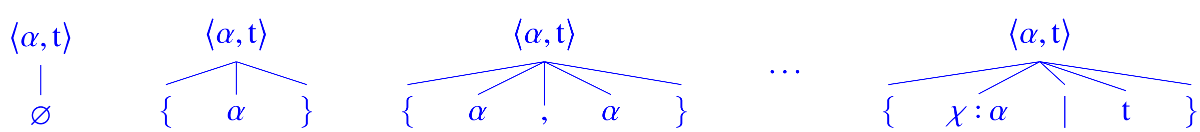
   for example, an expression meaning *the set of things Asia contains*:

$$[\![\{x{:}\mathrm{e} \mid \mathsf{Contain}\ x\ \mathsf{Asia}\}]\!]^M = [\![\lambda_{x:\mathrm{e}}\ \mathsf{Contain}\ x\ \mathsf{Asia}]\!]^M$$

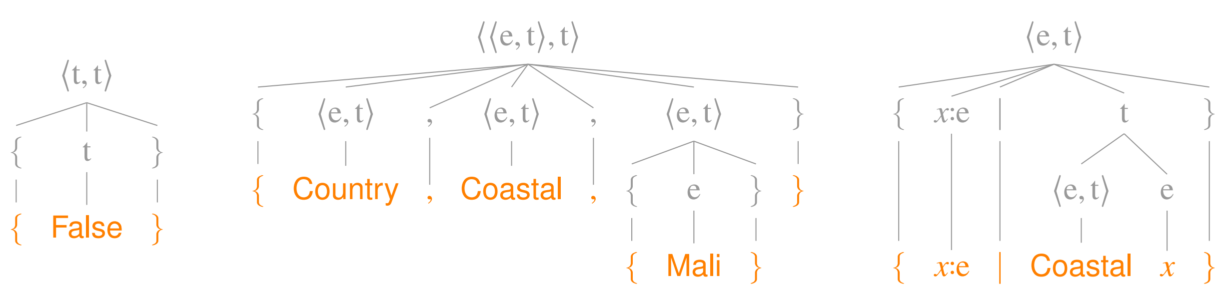
3. **null set** — type $\langle \alpha, \mathrm{t} \rangle$

$$[\![\varnothing]\!]^M = [\![\lambda_{x:\alpha}\ \mathsf{False}]\!]^M$$

We can draw derivation trees for these expressions in circumfix notation using flattened rules:



Note that we can have sets of any type (e.g. $\langle\langle \mathrm{e}, \mathrm{t}\rangle, \mathrm{t}\rangle$), since α can be any type (e.g. $\alpha = \langle \mathrm{e}, \mathrm{t}\rangle$):



We also have relations between elements of type α and sets of type $\langle\alpha, \mathrm{t}\rangle$:

4. **element** — type $\langle\alpha, \langle\langle\alpha, \mathrm{t}\rangle, \mathrm{t}\rangle\rangle$, true if element x satisfies the characteristic function of set s:

$$[\![x \in s]\!]^M = [\![s\ x]\!]^M$$

for example, *Mali is an element of the set containing Laos, Mali and Togo*:

$$[\![\mathsf{Mali} \in \{\mathsf{Laos}, \mathsf{Mali}, \mathsf{Togo}\}]\!]^M = [\![(\lambda_{x:\mathrm{e}}\ x = \mathsf{Laos} \vee x = \mathsf{Mali} \vee x = \mathsf{Togo})\ \mathsf{Mali}]\!]^M = \textbf{True}$$

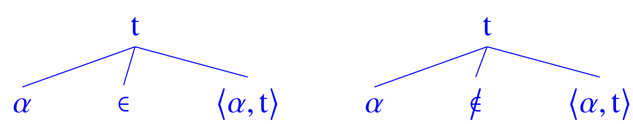
5. **negated element** — type $\langle\alpha, \langle\langle\alpha, \mathrm{t}\rangle, \mathrm{t}\rangle\rangle$, true if x does not satisfy s:

$$[\![x \notin s]\!]^M = [\![\mathsf{Not}\ (s\ x)]\!]^M$$

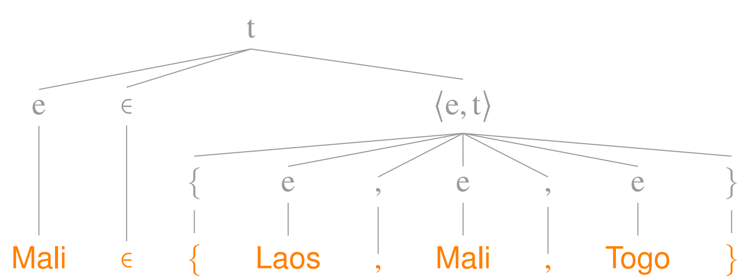
for example, *Peru is not an element of the set containing Laos, Mali and Togo*:

$$[\![\mathsf{Peru} \notin \{\mathsf{Laos}, \mathsf{Mali}, \mathsf{Togo}\}]\!]^M = [\![\mathsf{Not}\ ((\lambda_{x:\mathrm{e}}\ x = \mathsf{Laos} \vee x = \mathsf{Mali} \vee x = \mathsf{Togo})\ \mathsf{Peru})]\!]^M = \textbf{True}$$

We can draw derivation trees for these expressions in infix notation using flattened rules:



For example:



We also have relations between sets of type $\langle\alpha, \mathrm{t}\rangle$:

6. **subset or equal** — type $\langle\langle\alpha, \mathrm{t}\rangle, \langle\langle\alpha, \mathrm{t}\rangle, \mathrm{t}\rangle\rangle$, true if all elements of r are elements of s:

$$[\![r \subseteq s]\!]^M = [\![\forall_{x:\alpha}\ r\ x \rightarrow s\ x]\!]^M$$

for example, *the set of Laos and Mali is a subset of or equal to the set of Laos, Mali, Togo*:

$$\{\mathsf{Laos}, \mathsf{Mali}\} \subseteq \{\mathsf{Laos}, \mathsf{Mali}, \mathsf{Togo}\}$$

7. **superset or equal** — type $\langle\langle\alpha,\mathrm{t}\rangle,\langle\langle\alpha,\mathrm{t}\rangle,\mathrm{t}\rangle\rangle$, true if all elements of s are elements of r:

$$[\![r \supseteq s]\!]^M = [\![\forall_{x:\alpha}\ s\ x \rightarrow r\ x]\!]^M$$

for example, *the set of Laos, Mali, Togo is a superset of or equal to the set of Laos and Mali*:

$$\{\mathsf{Laos}, \mathsf{Mali}, \mathsf{Togo}\} \supseteq \{\mathsf{Laos}, \mathsf{Mali}\}$$

8. **proper subset/superset** — type $\langle\langle\alpha,\mathrm{t}\rangle,\langle\langle\alpha,\mathrm{t}\rangle,\mathrm{t}\rangle\rangle$, true if not equal and all in one are in other:

$$[\![r \subset s]\!]^M = [\![r \neq s \wedge \forall_{x:\alpha}\ r\ x \rightarrow s\ x]\!]^M$$
$$[\![r \supset s]\!]^M = [\![r \neq s \wedge \forall_{x:\alpha}\ s\ x \rightarrow r\ x]\!]^M$$

for example, *the set of Laos and Mali is a proper subset of the set of Laos, Mali and Togo*:

$$\{\mathsf{Laos}, \mathsf{Mali}\} \subset \{\mathsf{Laos}, \mathsf{Mali}, \mathsf{Togo}\}$$

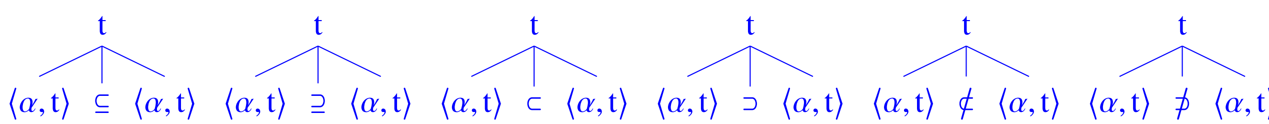
9. **negated subset/superset** — type $\langle\langle\alpha,\mathrm{t}\rangle,\langle\langle\alpha,\mathrm{t}\rangle,\mathrm{t}\rangle\rangle$, true if non-negated relation is false.

$$[\![r \not\subset s]\!]^M = [\![\neg\ (r \neq s \wedge \forall_{x:\alpha}\ r\ x \rightarrow s\ x)]\!]^M$$
$$[\![r \not\supset s]\!]^M = [\![\neg\ (r \neq s \wedge \forall_{x:\alpha}\ s\ x \rightarrow r\ x)]\!]^M$$

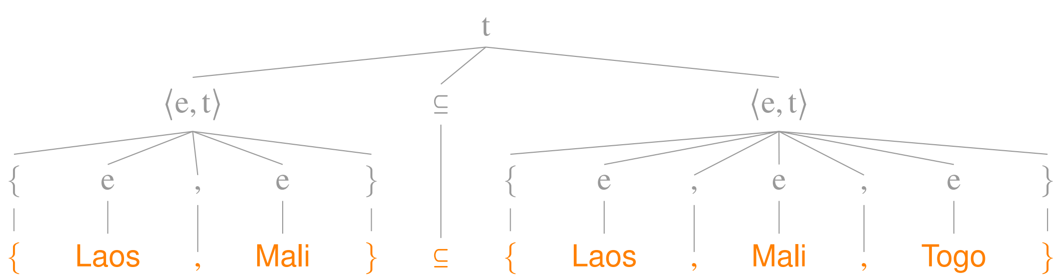
for example, *the set of Laos and Peru is not a proper subset of the set of Laos and Mali*:

$$\{\mathsf{Laos}, \mathsf{Peru}\} \not\subset \{\mathsf{Laos}, \mathsf{Mali}\}$$

We can draw derivation trees for these expressions in infix notation using flattened rules:



For example:



Practice 9.1:

Which of the following are true:

1. $\{\text{Mali}, \text{Togo}\} \subseteq \{\text{Mali}, \text{Togo}\}$
2. $\{\text{Mali}, \text{Togo}\} \not\subset \{\text{Mali}, \text{Togo}\}$
3. $\varnothing \in \{\text{Mali}, \text{Togo}\}$
4. $\varnothing \subset \{\text{Mali}, \text{Togo}\}$

We also have operators from sets of type $\langle\alpha, \mathrm{t}\rangle$ to sets of type $\langle\alpha, \mathrm{t}\rangle$:

10. **intersection** — type $\langle\langle\alpha, \mathrm{t}\rangle, \langle\langle\alpha, \mathrm{t}\rangle, \langle\alpha, \mathrm{t}\rangle\rangle\rangle$, outputs the set of entities in both r and s:

$$[\![r \cap s]\!]^M = [\![\lambda_{x:\alpha}\ r\ x \wedge s\ x]\!]^M$$

for example, *the intersection of the set of Laos and Mali with the set of Mali and Togo*:

$$\{\text{Laos}, \text{Mali}\} \cap \{\text{Mali}, \text{Togo}\} = \{\text{Mali}\}$$

11. **union** — type $\langle\langle\alpha, \mathrm{t}\rangle, \langle\langle\alpha, \mathrm{t}\rangle, \langle\alpha, \mathrm{t}\rangle\rangle\rangle$, outputs the set of entities in either r or s:

$$[\![r \cup s]\!]^M = [\![\lambda_{x:\alpha}\ r\ x \vee s\ x]\!]^M$$

for example, *the union of the set of Laos and Mali with the set of Mali and Togo*:

$$\{\text{Laos}, \text{Mali}\} \cup \{\text{Mali}, \text{Togo}\} = \{\text{Laos}, \text{Mali}, \text{Togo}\}$$

12. **exclusion** — type $\langle\langle\alpha, \mathrm{t}\rangle, \langle\langle\alpha, \mathrm{t}\rangle, \langle\alpha, \mathrm{t}\rangle\rangle\rangle$, outputs the set of entities in r but not in s:

$$[\![r - s]\!]^M = [\![\lambda_{x:\alpha}\ r\ x \wedge \neg(s\ x)]\!]^M$$

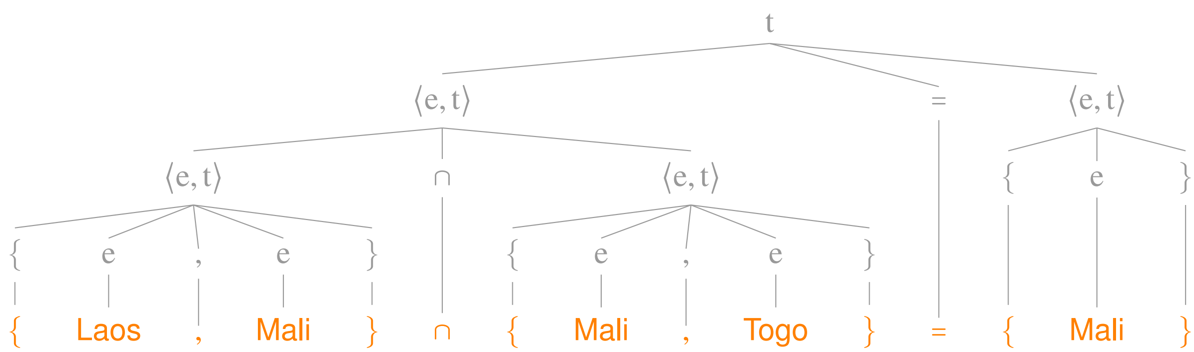
for example, *the set of Laos and Mali excluding the set of Mali and Togo*:

$$\{\text{Laos}, \text{Mali}\} - \{\text{Mali}, \text{Togo}\} = \{\text{Laos}\}$$

We can draw derivation trees for these expressions in infix notation using flattened rules:

$$\dfrac{\langle\alpha, \mathrm{t}\rangle \quad \cup \quad \langle\alpha, \mathrm{t}\rangle}{\langle\alpha, \mathrm{t}\rangle} \qquad \dfrac{\langle\alpha, \mathrm{t}\rangle \quad \cap \quad \langle\alpha, \mathrm{t}\rangle}{\langle\alpha, \mathrm{t}\rangle} \qquad \dfrac{\langle\alpha, \mathrm{t}\rangle \quad - \quad \langle\alpha, \mathrm{t}\rangle}{\langle\alpha, \mathrm{t}\rangle}$$

For example:



We also have functions from sets of type $\langle\alpha, \mathrm{t}\rangle$ to sets of these sets, of type $\langle\langle\alpha, \mathrm{t}\rangle, \mathrm{t}\rangle$:

13. **power set** — type $\langle\langle\alpha, \mathrm{t}\rangle, \mathrm{t}\rangle$, outputs the set of sets that are subsets of s:

$$[\![\mathcal{P}\ s]\!]^M = [\![\lambda_{r:\langle\alpha,\mathrm{t}\rangle}\ r \subseteq s]\!]^M$$

for example, *the power set of the set of Laos and Mali*:

$$[\![\mathcal{P}\ \{\text{Laos}, \text{Mali}\}]\!]^M = [\![\{\varnothing, \{\text{Laos}\}, \{\text{Mali}\}, \{\text{Laos}, \text{Mali}\}\}]\!]^M$$

Derivation trees for these expressions use ordinary function application.

Note that the power set is not a set of entities (which would be type $\langle\alpha, \mathrm{t}\rangle$).

If we want to intersect or subset this, we need to generalize the above operations.

We'll do that later, using 'rule schemas' ...

Practice 9.2:

Write an expression in ***set notation*** meaning *the set of all sets with no elements*.

Practice 9.3:

Write an expression in ***lambda calculus*** meaning *the set of all sets with no elements*.

References

[Cantor, 1874] Cantor, G. (1874). Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen. *Crelle's Journal für Mathematik*, 77, 258–263.

[Russell, 1902] Russell, B. (1902). Letter to frege. In Jean van Heijenoort (ed.), *From Frege to Gödel*, Cambridge, Mass.: Harvard University Press, 1967, 124–125.

[Zermelo, 1908] Zermelo, E. (1908). Untersuchungen über die Grundlagen der Mengenlehre. *Mathematische Annalen*, 65(2), 261–281. Der Artikel wurde am 30. Juli 1907 in Chesières fertiggestellt.