

Ling 3701H / Psych 3371H: Lecture Notes 8

Linear Algebra

8.1 Terms

We can define matrices and vectors as arrays of real numbers:

- s is a **scalar** iff $s \in \mathbb{R}$

You will often see scalars written as Greek letters, e.g.: γ

- \mathbf{v} is a **vector** iff $\mathbf{v} \in \mathbb{R}^I$

Scalars in vectors can be identified by one index: say $\mathbf{v} = \begin{bmatrix} 1.8 \\ -3 \end{bmatrix}$ then: $\mathbf{v}_{[2]} = -3$

- \mathbf{M} is a **matrix** iff $\mathbf{M} \in \mathbb{R}^{I \times J}$

Scalars in matrices can be identified by two indices: say $\mathbf{M} = \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix}$ then: $\mathbf{M}_{[2,1]} = -3$

8.2 Operations

- **transpose**: for all $\mathbf{M} \in \mathbb{R}^{I \times J}$, and all i, j indices to matrix rows and columns,

$$(\mathbf{M}^T)_{[i,j]} = \mathbf{M}_{[j,i]}$$

For example: $\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix}^T = \begin{bmatrix} 1.8 & -3 \\ 12 & 40 \end{bmatrix}$

- **scalar sum**: for all $s \in \mathbb{R}$, $\mathbf{M} \in \mathbb{R}^{I \times J}$, and all i, j indices to matrix rows and columns,

$$(s + \mathbf{M})_{[i,j]} = (\mathbf{M} + s)_{[i,j]} = s + \mathbf{M}_{[i,j]}$$

(commutative)

For example: $2 + \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} = \begin{bmatrix} 3.8 & 14 \\ -1 & 42 \end{bmatrix}$

- **matrix/vector sum**: for all $\mathbf{M}, \mathbf{N} \in \mathbb{R}^{I \times J}$, with row and column indices i, j ,

$$(\mathbf{M} + \mathbf{N})_{[i,j]} = (\mathbf{N} + \mathbf{M})_{[i,j]} = \mathbf{M}_{[i,j]} + \mathbf{N}_{[i,j]}$$

(commutative)

For example: $\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2.8 & 14 \\ 0 & 44 \end{bmatrix}$

- **scalar product**: for all $s \in \mathbb{R}$, $\mathbf{M} \in \mathbb{R}^{I \times J}$, with row and column indices i, j ,

$$(s \mathbf{M})_{[i,j]} = (\mathbf{M} s)_{[i,j]} = s \cdot \mathbf{M}_{[i,j]}$$

(commutative)

For example: $2 \begin{bmatrix} 1.8 & 12 \\ -3 & 40 \end{bmatrix} = \begin{bmatrix} 3.6 & 24 \\ -6 & 80 \end{bmatrix}$

- **matrix/vector product:** for all $\mathbf{M} \in \mathbb{R}^{I \times K}$, $\mathbf{N} \in \mathbb{R}^{K \times J}$, with indices i, j, k ,

$$(\mathbf{M}\mathbf{N})_{[i,j]} = \sum_k \mathbf{M}_{[i,k]} \cdot \mathbf{N}_{[k,j]}$$

(not commutative)

For example:

$$\begin{bmatrix} 1.8 & 12 \\ -3 & 40 \\ 15 & -6 \\ 7 & 18 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} (1.8 \cdot 1) + (12 \cdot 4) & (1.8 \cdot 2) + (12 \cdot 5) & (1.8 \cdot 3) + (12 \cdot 6) \\ (-3 \cdot 1) + (40 \cdot 4) & (-3 \cdot 2) + (40 \cdot 5) & (-3 \cdot 3) + (40 \cdot 6) \\ (15 \cdot 1) + (-6 \cdot 4) & (15 \cdot 2) + (-6 \cdot 5) & (15 \cdot 3) + (-6 \cdot 6) \\ (7 \cdot 1) + (18 \cdot 4) & (7 \cdot 2) + (18 \cdot 5) & (7 \cdot 3) + (18 \cdot 6) \end{bmatrix}$$

$$= \begin{bmatrix} 49.8 & 63.6 & 77.4 \\ 157 & 194 & 231 \\ -9 & 0 & 9 \\ 79 & 104 & 129 \end{bmatrix}$$

There are two special cases of matrix multiplication for vectors:

1. **inner ('dot') product:** for vectors $\mathbf{v}, \mathbf{u} \in \mathbb{R}^I$,

$$\mathbf{v}^\top \mathbf{u} = \sum_i \mathbf{v}_{[i]} \cdot \mathbf{u}_{[i]}$$

For example:

$$\begin{bmatrix} 1.8 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = (1.8 \cdot 1) + (-3 \cdot 2)$$

$$= -4.2$$

2. **outer product:** for vectors $\mathbf{v} \in \mathbb{R}^I$, $\mathbf{u} \in \mathbb{R}^J$,

$$(\mathbf{v}\mathbf{u}^\top)_{[i,j]} = \mathbf{v}_{[i]} \cdot \mathbf{u}_{[j]}$$

For example:

$$\begin{bmatrix} 1.8 \\ -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1.8 \cdot 1 & 1.8 \cdot 2 & 1.8 \cdot 3 \\ -3 \cdot 1 & -3 \cdot 2 & -3 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1.8 & 3.6 & 5.4 \\ -3 & -6 & -9 \end{bmatrix}$$

- **Kronecker product:** for all $\mathbf{M} \in \mathbb{R}^{I \times J}$, $\mathbf{N} \in \mathbb{R}^{K \times L}$,

$$\mathbf{M} \otimes \mathbf{N} = \begin{bmatrix} \mathbf{M}_{[1,1]} \mathbf{N} & \cdots & \mathbf{M}_{[1,J]} \mathbf{N} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{[I,1]} \mathbf{N} & \cdots & \mathbf{M}_{[I,J]} \mathbf{N} \end{bmatrix}$$

For example:

$$\begin{bmatrix} 1.1 & 4 \\ -3 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1.1 \cdot 1 & 1.1 \cdot 2 & 1.1 \cdot 3 & 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 1.1 \cdot 4 & 1.1 \cdot 5 & 1.1 \cdot 6 & 4 \cdot 4 & 4 \cdot 5 & 4 \cdot 6 \\ -3 \cdot 1 & -3 \cdot 2 & -3 \cdot 3 & 1 \cdot 1 & 1 \cdot 2 & 1 \cdot 3 \\ -3 \cdot 4 & -3 \cdot 5 & -3 \cdot 6 & 1 \cdot 4 & 1 \cdot 5 & 1 \cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1 & 2.2 & 3.3 & 4 & 8 & 12 \\ 4.4 & 5.5 & 6.6 & 16 & 20 & 24 \\ -3 & -6 & -9 & 1 & 2 & 3 \\ -12 & -15 & -18 & 4 & 5 & 6 \end{bmatrix}$$