

Generalized Entailment and Semantic Consequence

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(1) Generalized Entailment

Suppose $\langle P, \sqsubseteq, \sqcap, \top \rangle$ is a lower presemilattice with top. (Intuitively, think of the members as propositions, \sqsubseteq as entailment, \sqcap as the interpretation of *and*, and \top as some necessary truth.)

- a. We define **generalized entailment**, also denoted by \sqsubseteq , to be the following binary relation between $\wp(P)$ and P : for $S \subseteq P$ and $p \in P$,

$$S \sqsubseteq p \text{ iff } p \in \text{UB}(LB(S))$$

- b. Intuitively: a proposition p is entailed by a set of propositions S iff, for every way things might be, if every member of S is true with things that way, then so is p .
- c. Note that if S is finite, then $S \sqsubseteq p$ iff $q \sqsubseteq p$ where q is any of the (equivalent) glb's of S . (Note that the ambiguity of the symbol ' \sqsubseteq ' is disambiguated depending on whether a member of P or a subset of P is on the left-hand side of the inequality.)

(2) Notation for Generalized Entailment

We notate $S \sqsubseteq p$ as:

- a. $\sqsubseteq p$, if $S = \emptyset$
- b. $q \sqsubseteq p$, if $S = \{q\}$
Note that this is consistent with the notation for the case where \sqsubseteq denotes a binary relation on P .
- c. $p_1, \dots, p_n \sqsubseteq p$, if $S = \{p_1, \dots, p_n\}$
Note that in this case:
 - i. The order in which the elements of S are listed on the left-hand side is irrelevant.
 - ii. Repetitions on the left-hand side are irrelevant.

(3) **Simple Observations about Generalized Entailment**

- a. $\sqsubseteq p$ iff $p \equiv \top$ (i.e. p is a top)
- b. $p, q \sqsubseteq r$ iff $p \sqcap q \sqsubseteq r$ more generally
- c. If additionally P has an rpc operation \rightarrow , then $r \sqsubseteq p \rightarrow q$ iff $r, p \sqsubseteq q$ iff $r \sqcap p \sqsubseteq q$.

(4) **HPS Interpretation (Review)**

Let Φ be the set of PIPL formulas over a fixed finite set of propositional letters. Then recall that an **HPS interpretation of PIPL** is a heyting presemilattice (HPS) $\langle P, \sqsubseteq, \top, \sqcap, \rightarrow \rangle$. together with a function **sem** from Φ to P that satisfies the following conditions, for all formulas ϕ and ψ :

- i. $\text{sem}(\top) = \top$
- ii. $\text{sem}(\phi \wedge \psi) = \text{sem}(\phi) \sqcap \text{sem}(\psi)$
- iii. $\text{sem}(\phi \rightarrow \psi) = \text{sem}(\phi) \rightarrow \text{sem}(\psi)$

(5) **Semantic Consequence**

- a. By a **context**, we mean a finite set of formulas.
- b. We define a binary relation between contexts and formulas called **semantic consequence**, denoted by \models , as follows:
 $\Gamma \models \psi$ iff, for every HPS interpretation, $\text{sem}[\Gamma] \sqsubseteq \text{sem}(\psi)$.

(6) **Notation in Assertions of Semantic Consequence**

- a. We use Γ and Δ as metavariables over contexts.
- b. A singleton context $\{\phi\}$ is abbreviated as ϕ , so $\phi \models \psi$ abbreviates $\{\phi\} \vdash \psi$.
- c. The empty context is abbreviated by writing nothing, so $\models \psi$ abbreviates $\emptyset \vdash \psi$.
- d. Contexts with more than one member are abbreviated by eliminating the set braces, so $\phi_1, \dots, \phi_n \models \psi$ abbreviates $\{\phi_1, \dots, \phi_n\} \vdash \psi$.
- e. Γ, Δ abbreviates $\Gamma \cup \Delta$.
- f. Γ, ϕ abbreviates $\Gamma \cup \{\phi\}$.

(7) **Some Simple Observations about Semantic Consequence**

- a. $\models \psi$ iff $\text{sem}(\psi) \equiv \top$ for every HPS interpretation.
- b. $\phi \models \psi$ iff $\text{sem}(\phi) \sqsubseteq \text{sem}(\psi)$ for every HPS interpretation.
- c. $\phi_1, \dots, \phi_n \models \psi$ iff $\text{sem}(\phi_1), \dots, \text{sem}(\phi_n) \sqsubseteq \text{sem}(\psi)$ for every HPS interpretation.