Generalized Entailment and Semantic Consequence

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These slides are available at:

`http://www.ling.ohio-state.edu/~plummer/ling681`

(1) **Generalized Entailment**

Suppose $\langle P, \sqsubseteq, \sqcap, \top \rangle$ is a lower presemilattice with top. (Intuitively, think of the members as propositions, $\sqsubseteq$ as entailment, $\sqcap$ as the interpretation of *and*, and $\top$ as some necessary truth.)

a. We define **generalized entailment**, also denoted by $\sqsubseteq$, to be the following binary relation between $\wp(P)$ and P: for $S \subseteq P$ and $p \in P$,

$$S \sqsubseteq p \text{ iff } p \in \mathrm{UB}(LB(S))$$

b. Intuitively: a proposition p is entailed by a set of propositions S iff, for every way things might be, if every member of S is true with things that way, then so is p.

c. Note that if S is finite, then $S \sqsubseteq p$ iff $q \sqsubseteq p$ where q is any of the (equivalent) glb's of S. (Note that the ambiguity of the symbol '$\sqsubseteq$' is disambiguated depending on whether a member of P or a subset of P is on the left-hand side of the inequality.)

(2) **Notation for Generalized Entailment**

We notate $S \sqsubseteq p$ as:

a. $\sqsubseteq p$, if $S = \emptyset$

b. $q \sqsubseteq p$, if $S = \{q\}$
Note that this is consistent with the notation for the case where $\sqsubseteq$ denotes a binary relation on P.

c. $p_1, \ldots, p_n \sqsubseteq p$, if $S = \{p_1, \ldots, p_n\}$
Note that in this case:

  i. The order in which the elements of S are listed on the left-hand side is irrelevant.

  ii. Repetitions on the left-hand side are irrelevant.

(3) **Simple Observations about Generalized Entailment**

a. $\sqsubseteq p$ iff $p \equiv \top$ (i.e. p is a top

b. $p, q \sqsubseteq r$ iff $p \sqcap q \sqsubseteq r$ more generally

c. If additionally P has an rpc operation $\rightarrow$, then $r \sqsubseteq p \rightarrow q$ iff $r, p \sqsubseteq q$ iff $r \sqcap p \sqsubseteq q$.

(4) **HPS Interpretation (Review)**

Let Φ be the set of PIPL formulas over a fixed finite set of propositional letters. Then recall that an **HPS interpretation of PIPL** is a heyting presemilattice (HPS) $\langle P, \sqsubseteq, \top, \sqcap, \rightarrow \rangle$. together with a function sem from Φ to P that satisfies the following conditions, for all formulas ϕ and ψ:

i. $\mathsf{sem}(T) = \top$

ii. $\mathsf{sem}(\phi \wedge \psi) = \mathsf{sem}(\phi) \sqcap \mathsf{sem}(\psi)$

iii. $\mathsf{sem}(\phi \rightarrow \psi) = \mathsf{sem}(\phi) \rightarrow \mathsf{sem}(\psi)$

(5) **Semantic Consequence**

a. By a **context**, we mean a finite set of formulas.

b. We define a binary relation between contexts and formulas called **semantic consequence**, denoted by $\models$, as follows:
$\Gamma \models \psi$ iff, for every HPS interpretation, $\mathsf{sem}[\Gamma] \sqsubseteq \mathsf{sem}(\psi)$.

(6) **Notation in Assertions of Semantic Consequence**

a. We use Γ and Δ as metavariables over contexts.

b. A singleton context $\{\phi\}$ is abbreviated as ϕ, so $\phi \models \psi$ abbreviates $\{\phi\} \vdash \psi$.

c. The empty context is abbreviated by writing nothing, so $\models \psi$ abbreviates $\emptyset \vdash \psi$.

d. Contexts with more than one member are abbreviated by eliminating the set braces, so $\phi_1, \ldots, \phi_n \models \psi$ abbreviates $\{\phi_1, \ldots, \phi_n\} \models \psi$.

e. Γ, Δ abbreviates $\Gamma \cup \Delta$.

f. Γ, ϕ abbreviates $\Gamma \cup \{\phi\}$.

(7) **Some Simple Observations about Semantic Consequence**

a. $\models \psi$ iff $\mathsf{sem}(\psi) \equiv \top$ for every HPS interpretation.

b. $\phi \models \psi$ iff $\mathsf{sem}(\phi) \sqsubseteq \mathsf{sem}(\psi)$ for every HPS interpretation.

c. $\phi_1, \ldots, \phi_n \models \psi$ iff $\mathsf{sem}(\phi_1), \ldots, \mathsf{sem}(\phi_n) \sqsubseteq \mathsf{sem}(\psi)$ for every HPS interpretation.