(Pre-)(semi-)lattices

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(1) **Meet and join operations**

Suppose $\sqsubseteq$ is a preorder on A A binary operation $\sqcap$ ($\sqcup$) on A is called a **meet (join)** operation provided, for all $a, b \in A$, $a \sqcap b$ ($a \sqcup b$) is a glb (lub) of $\{a, b\}$ (not necessarily the only one).

It follows from the definition of glb that, for all $a, b, c \in A$:

a. (Meet Elimination 1) $a \sqcap b \sqsubseteq a$;

b. (Meet Elimination 2) $a \sqcap b \sqsubseteq b$; and

c. (Meet Introduction) if $c \sqsubseteq a$ and $c \sqsubseteq b$, then $c \sqsubseteq a \sqcap b$.

It follows from the definition of lub that, for all $a, b, c \in A$:

d. (Join Introduction 1) $a \sqsubseteq a \sqcup b$;

e. (Join Introduction 2) $b \sqsubseteq a \sqcup b$; and

f. (Join Elimination) if $a \sqsubseteq c$ and $b \sqsubseteq c$, then $a \sqcup b \sqsubseteq c$.

(2) **(Pre-)(semi-)lattices**

a. A preordered set equipped with a meet (join) operation is a called a **lower (upper) presemilattice**, and one equipped with both is called a **prelattice**.

b. A presemilattice (prelattice) whose underlying preorder is an order is called a **semilattice (lattice)**.

(3) **Facts about $\sqcap$ ($\sqcup$) in a lower (upper) presemilattice**

Here 'u.t.e.' abbreviates 'up to equivalence'.

a. (Idempotence u.t.e.) $a \sqcap a \equiv a$;

b. (Commutativity u.t.e.) $a \sqcap b \equiv b \sqcap a$;

c. (Associativity u.t.e.) $(a \sqcap b) \sqcap c \equiv a \sqcap (b \sqcap c)$;

d. (Monotonicity on Both Sides) For each $a \in A$, the function that maps each $b \in A$ to $a \sqcap b$ ($b \sqcap a$) is monotonic.

e. (Substititivity u.t.e.) If $a \equiv c$ and $b \equiv d$ then $a \sqcap b \equiv c \sqcap d$.

f. The preceding assertions remain true if $\sqcap$ is replaced by $\sqcup$.

g. (Interdefinability) $a \sqsubseteq b$ iff $a \sqcap b \equiv a$ (iff $a \sqcup b \equiv b$).

h. If the preorder is an order, then all occurrences of $\equiv$ in the preceding can be replace by $=$.

(4) **Facts about $\sqcap$ and $\sqcup$ in a prelattice**

a. (Absorption u.t.e.) $(a \sqcup b) \sqcap b \equiv b \equiv (a \sqcap b) \sqcup b$;

b. (Semidistributivity) $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup c)$.

(5) **Distributive prelattices**

a. A prelattice is called **distributive** if the inequality reverse to Semidistributivity holds: $a \sqcap (b \sqcup c) \sqsubseteq (a \sqcap b) \sqcup (a \sqcap c)$. holds.

b. Thus in a distributive prelattice, we have the following (Distributivity u.t.e.): $a \sqcap (b \sqcup c) \equiv (a \sqcap b) \sqcup (a \sqcap c)$.

c. It can be shown that this equivalence holds in a prelattice just in case the dual one (formed by interchanging meets and joins) does.

(6) **Relative pseudocomplement (rpc)**

Suppose $\langle A, \sqsubseteq, \sqcap \rangle$ is a lower presemilattice.

a. If $a, b, c \in A$, then c is called a **relative pseudocomplement (rpc)** of a **relative to** b iff the following two conditions hold:

   i. $c \sqcap a \sqsubseteq b$; and

   ii. for all $d \in A$, if $d \sqcap a \sqsubseteq b$, then $d \sqsubseteq c$.

b. Equivalently, c is an rpc of a relative to b iff it is a greatest member of $\{x \in A \mid x \sqcap a \sqsubseteq b\}$.

c. It follows that all rpc's of a relative to b are equivalent.

(7) **RPC Operations**

A binary operation $\to$ on A is called an **rpc operation** iff, for all $a, b \in A$, $a \to b$ is an rpc of a relative to b.

It follows from the defining conditions (6) for an rpc that:

a. (RPC Elimination) $(a \to b) \sqcap a \sqsubseteq b$; and

b. (RPC Introduction) if $c \sqcap a \sqsubseteq b$, then $c \sqsubseteq a \to b$.

Other important properties of rpc operations include these:

c. (Converse of RPC Introduction) if $c \sqsubseteq a \to b$, then $c \sqcap a \sqsubseteq b$.

d. (Antitonicity in 1st argument) if $a \sqsubseteq b$ then $(b \to c) \sqsubseteq (a \to c)$.

e. (Monotonicity in 2nd argument) if $a \sqsubseteq b$, then $(c \to a) \sqsubseteq (c \to b)$.

f. (Substititivity u.t.e.) If $a \equiv c$ and $b \equiv d$ then $a \to b \equiv c \to d$.

(8) **Heyting (pre-)semilattices**

A lower presemilattice $\langle A, \sqsubseteq, \sqcap \rangle$ equipped with a top $\top$ and an rpc operation $\to$ is called a **heyting presemilattice**, and a **heyting semilattice** if $\sqsubseteq$ is an order.

Some facts about heyting presemilattices:

a. $a \sqcap \top \equiv a$

b. $a \to a \equiv \top$

c. $a \sqcap (a \to b) \equiv a \sqcap b$

d. $(a \to b) \sqcap b \equiv b$

e. $a \to (b \sqcap c) \equiv (a \to b) \sqcap (a \to c)$

f. $(a \to b) \sqcap (b \to c) \sqsubseteq a \to c$.

g. $a \sqsubseteq b$ iff $a \to b \equiv \top$.