

Review of Preorders

Carl Pollard
Ohio State University

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(1) Definitions

- a. A **preorder** on a set A is a binary relation \sqsubseteq (read ‘less than or equivalent to’) on A which is reflexive and transitive.
- b. An antisymmetric preorder is called an **order**.
- c. The equivalence relation \equiv **induced** by the preorder is defined by $a \equiv b$ iff $a \sqsubseteq b$ and $b \sqsubseteq a$. (Note that, if \sqsubseteq is an order, this reduces to the identity relation on A , and correspondingly \sqsubseteq is read as ‘less than or equal to’.)

(2) Background Assumptions

Until further notice:

- a. \sqsubseteq is a preorder on A
- b. \equiv is the equivalence relation it induces
- c. $S \subseteq A$
- d. $a \in A$

(3) Definitions

- a. We call a an **upper (lower) bound** of S iff, for every $b \in S$, $b \sqsubseteq a$ ($a \sqsubseteq b$).

Suppose moreover that $a \in S$. Then a is said to be:

- b. **greatest (least)** in S iff it is an upper (lower) bound of S ;
- c. a **top (bottom)** iff it is greatest (least) in A ;
- d. **maximal (minimal)** in S iff, for every $b \in S$, if $a \sqsubseteq b$ ($b \sqsubseteq a$), then $b \sqsubseteq a$ ($a \sqsubseteq b$), so that in fact $a \equiv b$.

(4) **Observations**

- a. If a is greatest (least) in S , then it is maximal (minimal) in S .
- b. All greatest (least) members of S are equivalent.
- c. So if \sqsubseteq is an order, S has at most one greatest (least) member, and A has at most one top (bottom).

(5) **Definitions**

Let $UB(S)$ ($LB(S)$) be the set of upper (lower) bounds of S . (Note: if $S = \{a\}$, then $UB(S)$ ($LB(S)$) is usually written $\uparrow a$ ($\downarrow a$), read ‘up of a ’ (‘down of a ’)).

- a. A least member of $UB(S)$ is called a **least upper bound (lub)** of S .
- b. A greatest member of $LB(S)$ is called a **greatest lower bound (glb)** of S .

(6) **Observations**

- a. Any greatest (least) member of S is a lub (glb) of S .
- b. All lubs (glbs) of S are equivalent.
- c. If \sqsubseteq is an order, then S has at most one lub (glb).
- d. A lub (glb) of A is the same thing as a top (bottom).
- e. A lub (glb) of \emptyset is the same thing as a bottom (top).

(7) **Binary glbs and lubs**

If $S = \{a, b\}$ and S has a unique glb (lub), it is written $a \sqcap b$ ($a \sqcup b$).

(8) **Facts about \sqcap and \sqcup when \sqsubseteq is an order**

- a. (Idempotence) $a \sqcap a$ exists and equals a .
- b. (Commutativity) If $a \sqcap b$ exists, so does $b \sqcap a$, and they are equal.
- c. (Associativity) If $(a \sqcap b) \sqcap c$ and $a \sqcap (b \sqcap c)$ both exist, they are equal.

- d. The preceding three assertions remain true if \sqcap is replaced by \sqcup .
- e. (Interdefinability) $a \sqsubseteq b$ iff $a \sqcap b$ exists and equals a iff $a \sqcup b$ exists and equals b .
- f. (Absorbtion)
 - i. If $(a \sqcap b) \sqcup b$ exists, it equals b .
 - ii. If $(a \sqcup b) \sqcap b$ exists, it equals b .

(9) **Definitions**

- a. Suppose A and B are preordered by \sqsubseteq and \leq respectively. Then a function $f: A \rightarrow B$ is called:
 - i. **monotonic** or **order-preserving** iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a) \leq f(a')$;
 - ii. **antitonic** or **order-reversing** iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a') \leq f(a)$.
- b. A monotonic (antitonic) bijection is called a **preorder isomorphism (preorder anti-isomorphism)** provided its inverse is also monotonic (antitonic).
- c. Two preordered sets are said to be **preorder-isomorphic** provided there is a preorder isomorphism from one to the other.