REVIEW OF PREORDERS

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These slides are available at:

http://www.ling.ohio-state.edu/∼plummer/ling681

(1) **Definitions**

a. A **preorder** on a set A is a binary relation $\sqsubseteq$ (read 'less than or equivalent to') on A which is reflexive and transitive.

b. An antisymmetric preorder is called an **order**.

c. The equivalence relation $\equiv$ **induced** by the preorder is defined by $a \equiv b$ iff $a \sqsubseteq b$ and $b \sqsubseteq a$. (Note that, if $\sqsubseteq$ is an order, this reduces to the identity relation on A, and correspondingly $\sqsubseteq$ is read as 'less than or equal to'.)

(2) **Background Assumptions**

Until further notice:

a. $\sqsubseteq$ is a preorder on A

b. $\equiv$ is the equivalence relation it induces

c. $S \subseteq A$

d. $a \in A$

(3) **Definitions**

a. We call a an **upper** (**lower**) **bound** of S iff, for every $b \in S$, $b \sqsubseteq a$ ($a \sqsubseteq b$).

Suppose moreover that $a \in S$. Then a is said to be:

b. **greatest** (**least**) in S iff it is an upper (lower) bound of S;

c. a **top** (**bottom**) iff it is greatest (least) in A;

d. **maximal** (**minimal**) in S iff, for every $b \in S$, if $a \sqsubseteq b$ ($b \sqsubseteq a$), then $b \sqsubseteq a$ ($a \sqsubseteq b$), so that in fact $a \equiv b$.

(4) **Observations**

a. If a is greatest (least) in S, then it is maximal (minimal) in S.

b. All greatest (least) members of S are equivalent.

c. So if $\sqsubseteq$ is an order, S has at most one greatest (least) member, and A has at most one top (bottom).

(5) **Definitions**

Let UB(S) (LB(S)) be the set of upper (lower) bounds of S. (Note: if $S = \{a\}$, then UB(S) (LB(S)) is usually written $\uparrow a$ ($\downarrow a$), read 'up of a' ('down of a')).

a. A least member of UB(S) is called a **least upper bound (lub)** of S.

b. A greatest member of LB(S) is called a **greatest lower bound (glb)** of S.

(6) **Observations**

a. Any greatest (least) member of S is a lub (glb) of S.

b. All lubs (glbs) of S are equivalent.

c. If $\sqsubseteq$ is an order, then S has at most one lub (glb).

d. A lub (glb) of A is the same thing as a top (bottom).

e. A lub (glb) of $\emptyset$ is the same thing as a bottom (top).

(7) **Binary glbs and lubs**

If $S = \{a, b\}$ and S has a unique glb (lub), it is written $a \sqcap b$ ($a \sqcup b$).

(8) **Facts about $\sqcap$ and $\sqcup$ when $\sqsubseteq$ is an order**

a. (Idempotence) $a \sqcap a$ exists and equals a.

b. (Commutativity)If $a \sqcap b$ exists, so does $b \sqcap a$, and they are equal.

c. (Associativity) If $(a \sqcap b) \sqcap c$ and $a \sqcap (b \sqcap c)$ both exist, they are equal.

d. The preceding three assertions remain true if $\sqcap$ is replaced by $\sqcup$.

e. (Interdefinability) $a \sqsubseteq b$ iff $a \sqcap b$ exists and equals a iff $a \sqcup b$ exists and equals b.

f. (Absorbtion)

   i. If $(a \sqcap b) \sqcup b$ exists, it equals b.

   ii. If $(a \sqcup b) \sqcap b$ exists, it equals b.

(9) **Definitions**

a. Suppose A and B are preordered by $\sqsubseteq$ and $\leq$ respectively. Then a function $f\colon A \to B$ is called:

   i. **monotonic** or **order-preserving** iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a) \leq f(a')$;

   ii. **antitonic** or **order-reversing** iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a') \leq f(a)$.

b. A monotonic (antitonic) bijection is called a **preorder isomorphism (preorder anti-isomorphism)** provided its inverse is also monotonic (antitonic).

c. Two preordered sets are said to be **preorder-isomorphic** provided there is a preorder isomorphism from one to the other.