Positive Intuitionistic Propositional Logic (PIPL) and its Algebraic Semantics

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These slides are available at:

http://www.ling.ohio-state.edu/∼plummer/ling681

(1) **Why PIPL?**

- PIPL is a nice middle-of-the-road logic, in the sense that
  - the most standard (intuitionistic and classical) propositional logics are easily obtained from it by adding disjunction and negation (Chapter 11), while
  - the linguistically useful *substructural* logics that have proliferated in recent decades are easily obtained from it by removing inference rules (Chapter 14).
- PIPL also provides the type logic that underlies the all-important *typed lambda calculus* (Chapters 12 and 13).

(2) **PIPL as a formal language**

- We start with a finite set Let of *(propositional) letters*, the *nullary connective* T (read 'true'), the two *binary connectives* $\wedge$ 'and' and $\rightarrow$ 'implies', and the two *auxiliary symbols* ('left paren' and) 'right paren'.
- We use upper-case italic letters from the end of the Roman alphabet as metavariables over propositional letters.
- The PIPL language is a certain set of strings over the alphabet $\mathsf{Let} \cup \{T, \wedge, \rightarrow, (,)\}$.
- The strings in the PIPL language are called (PIPL) **formulas**.
- We will use certain lower-case Greek letters (especially ϕ, ψ, ξ, and ζ) as metavariables over formulas.

(3) **Definition of the set of PIPL formulas**

- Pedantic version:

  a. Each (length-one string consisting of just a single) propositional letter is a formula.

  b. The (length-one string consisting of just) T is a formula.

  c. If ϕ and ψ are formulas and c is a binary connective, then $(\phi\ \mathsf{c}\ \psi)$, i.e. the string consisting of (followed by the string ϕ followed by c followed by the string ψ followed by), is a formula.

- Less pedantic, more colloquial version:

  a. $X \in \Phi$ (for $X \in \mathsf{Let}$)

  b. $T \in \Phi$

  c. $(\phi\ \mathsf{c}\ \psi) \in \Phi$ (for $\phi, \psi \in \Phi$ and $\mathsf{c} \in \{\wedge, \rightarrow\}$).

(4) **Formulas are idealizations of declarative sentences**

- $\wedge$ and $\rightarrow$ correspond to the sentence conjunctions 'and' and 'if ... then' (or 'implies').
- Formulas (called **atomic**) containing no binary connectives correspond to 'simple' English declarative sentences (ones with no sentential conjunctions).
- T corresponds to a simple sentence which is *necessarly true* (true no matter how the world is).
- Other atomic formulas correspond to simple sentences which are *contingent* (true or false depending how the world is).
- Internal structure of simple sentences is disregarded.
- The possibility that there might be more than one necessarily true simple sentence is ignored.
- The distinction between a sentence and an utterance of that sentence is ignored.

(5) **Semantic Interpretation**

a. It is standard to assume that semantic interpretation is a function that maps (utterances of) linguistic expressions to the meanings they express.

b. Different kinds of linguistic expressions express different kinds of meanings.

c. The meanings expressed by declarative sentences are usually called **propositions**.

d. The meanings of the binary sentential connectives, such as *and* and *if* . . . *then* (or *implies*), are assumed to be binary operations on propositions.

(6) **Propositions**

a. Intuitively, propositions are things that are either true or false.

b. Propositions are called:

   i. **necessary truths** if they are true no matter how things are;

   ii. **necessary falsehoods** if they are false no matter how things are; and

   iii. **contingent** otherwise (i.e. their truth value depends on how things are).

c. There is usually assumed to be a binary relation on propositions called **entailment**. Intuitively, p entails q means that, no matter how things are, if p is true with things that way, then so is q.

d. From the preceding, it is easy to see that entailment is a preorder.

e. For two declarative sentences S and S′, we say S′ **follows from** S iff the proposition expressed by S entails the one expressed by S′.

(7) **Modelling Propositions**

a. We use a preordered set to model the set of propositions preordered by entailment.

b. Clearly any proposition entails any necessary truth, so we model necessary truths as tops.

c. Below we will show, based on how *and* and *if ... then* work in NL argumentation, that the binary relations that model their meanings should be, respectively, a meet operation and an rpc operation.

d. In short, we model propositions using a heyting presemilattice (hereafter, HPS).

e. Later (Chapter 11), we will add more structure to model the meanings of *or* and *it is not the case that*, obtaining a kind of preordered algebra called a *boolean prelattice*.

(8) **Modelling Semantic Interpretation**

Putting the pieces together:

a. We model NL declarative sentences by PIPL formulas.

b. We model propositions by an HPS.

c. We model semantic intepretation by an **HPS interpretation of PIPL**, which is defined to be a an HPS $\langle P, \sqsubseteq, \top, \sqcap, \rightarrow \rangle$. together with a function sem from Φ to P that satisfies the following conditions, for all formulas ϕ and ψ:

   i. $\mathsf{sem}(T) = \top$

   ii. $\mathsf{sem}(\phi \wedge \psi) = \mathsf{sem}(\phi) \sqcap \mathsf{sem}(\psi)$

   iii. $\mathsf{sem}(\phi \rightarrow \psi) = \mathsf{sem}(\phi) \rightarrow \mathsf{sem}(\psi)$

(9) **Why *and* is interpreted as a meet operation**

For any English sentences S, S', and S", according to the intuitions of native English speakers:

a. S' follows from the conjoined sentence S' and S".

b. S" follows from the conjoined sentence S' and S".

c. If S' and S" both follow from S, then so does the conjoined sentence S' and S".

(10) **Why *if* ... *then* is interpreted as an rpc operation**

For any English sentences S, S', and S", according to the intuitions of native English speakers:

a. S' follows from the conjoined sentence: if S then S', and S.

b. If S' follows from the conjoined sentence S" and S, then the conditional sentence if S then S' follows from S".