

CHAPTER NINE: PREORDERED ALGEBRA SEMANTICS FOR PROPOSITIONAL LOGIC

1 Introduction

The traditional valuation semantics for propositional logic (PL, Chapter 8) is a pretty blunt instrument for gaining insight into the connection between the form and meaning of natural language. There is no distinction between sentences and their contextually situated utterances. There is no articulation of the internal structure of atomic sentences (ones which contain no occurrences of the “logic words”). There is only one atomic sentence (T) that is necessarily true, and only one (F) that is necessarily false.

There are glaring inadequacies on the semantic side as well, especially from the generally Fregean perspective introduced in Chapter 8, section 1. Here we concern ourselves with just two. First, the “senses” are themselves set-theoretic entities—sets of valuations—constructed out of formulas. This is at odds with the view that the communicative function of language arises from the objective significance of meaning, or to put it another way, that semantic interpretation connects linguistic form to things (individuals, propositions, properties, etc.) whose existence is independent of language. And second, since entailment is modelled as the subset inclusion relation on sets of valuations, it follows that entailment is antisymmetric and therefore that formulas whose meanings include each other must mean the same thing. In natural language terms, this amounts to accepting the highly controversial position that truth-conditionally equivalent sentences express the same proposition.

Among linguistic semanticists, the usual way out of the first problem is to follow Montague, who in turn followed Kripke (1963) in modelling “ways things might be” not as linguistic constructs (PL valuations, or Carnapian state descriptions) but rather as theoretical primitives, usually called **possible worlds**.¹ On this approach, propositions are just sets of worlds and the usual logical connectives (or their natural-language counterparts) are interpreted as set-theoretic operations just as in the valuation semantics of PL, e.g. \wedge (or its English counterpart the sentential conjunction *and*) is interpreted as intersection of sets of possible worlds.

But the second problem remains untouched: entailment is still just set

¹Kripke introduced possible worlds as primitives in order to provide a semantics for modal logic, but possible worlds semantics still makes sense in the absence of explicit modal operators.

inclusion, and so sentences that are “true at the same worlds” (i.e. whose meanings have the same worlds as members) have the same proposition as their meanings. In this chapter we consider a different semantics, one whose historical roots predate Montague and Kripke.² On this approach, the basic idea is to take propositions themselves, rather than worlds, as the theoretical primitives, the entailment preorder as another primitive, and the meanings of the logical connectives as suitably well-behaved operations that turn the preorder into an algebra. (Possible worlds are still there, but not as theoretical primitives; instead, as we will see in a later chapter, *they* are constructed algebraically as *maximal consistent sets* of propositions, which we will define in due course).

For the time being, we’ll continue to model natural-language declarative sentences by PL formulas. In subsequent chapters, we will turn to the semantic interpretation of (fragments of) natural language, but for that we will first need to avail ourselves of models of natural-language syntax better suited to this task than the CFGs of Chapter 6.

Next, as discussed briefly in Chapter 3, section 3.2, we assume there is a set \mathbf{P} of things called **propositions**, with a preorder \sqsubseteq representing entailment, in which tops represent necessary truths, bottoms necessary falsehoods, and other elements contingent propositions. Additionally, we assume there is a (**semantic**) **interpretation** function \mathbf{sem} from formulas to propositions. We will say that ψ **follows from** ϕ (relative to the interpretation) iff $\mathbf{sem}(\phi) \sqsubseteq \mathbf{sem}(\psi)$.³

As in Chapter 7, we will denote the mutual entailment relation by \equiv . We assume also that there is a (not necessarily unique) proposition \top which is a top and another (not necessarily unique) proposition \perp which is a bottom. Clearly, if $\mathbf{sem}(\phi) \equiv \top$, then ϕ follows from every formula, and if $\mathbf{sem}(\phi) \equiv \perp$, then every formula follows from ϕ .

What about the semantic interpretation of the logical connectives? The general answer is that they will be certain operations on propositions. For expository convenience, we will introduce them one at a time, starting with conjunction (\wedge). This will be interpreted by a certain binary operation on propositions written \sqcap . That is, we will require of our interpretation function

²The underlying mathematics was worked out by various algebraists and logicians, including Marshall Stone, Alfred Tarski, and Bjarni Jónsson, in the 1930s and 1940s.

³Of course, in the case of natural language, we will want to arrange things in such a way that whenever we observe that native speakers judge sentence ψ to necessarily be true if sentence ϕ is (no matter how things might be), then ψ should also follow from ϕ in this technical sense. That is, we will want our semantics to be a good mathematical idealization of native intuitions about what follows from what.

sem, that, for any two formulas ϕ and ψ , $\text{sem}(\phi \wedge \psi) = \text{sem}(\phi) \sqcap \text{sem}(\psi)$.

2 Interpreting Conjunction

What kind of operations should \sqcap be? The basic generalizations about what follows from what, as far as \wedge is concerned, are that, for any ϕ , ψ , and ξ ,

ϕ follows from $\phi \wedge \psi$;

ψ follows from $\phi \wedge \psi$; and

if ϕ and ψ both follow from ξ , then so does $\phi \wedge \psi$.

In the case of natural language, with *and* in place of \wedge , these are empirical generalizations about how native English speakers judge the validity of certain kinds of simple arguments. In the setting of standard PL, the first two generalizations are called **conjunction eliminations**, because the conjunctions present in the premisses are missing from the conclusions; and the third is called **conjunction introduction**, because the conclusion introduces an occurrence of conjunction even though none are present in the premisses.

On a moment's reflection, it becomes clear that the most straightforward way to get our semantics to make the right predictions about how conjunction works in arguments is to require that \sqcap have the following properties, for all $p, q, r \in \mathbf{P}$:

$p \sqcap q \sqsubseteq p$;

$p \sqcap q \sqsubseteq q$; and

if $r \sqsubseteq p$ and $r \sqsubseteq q$, then $r \sqsubseteq p \sqcap q$

An operation on a preorder with these properties is precisely what we referred to (Chapter 7, section 2.2) as a **greatest lower bound (glb)** operation, and we called a preorder endowed with such an operation a **lower presemilattice**.

3 Preordered Algebras

We've already encountered the idea of an algebra as a set together with some operations. A **preordered algebra** is a set P together with a preorder \sqsubseteq

on P and a collection of operations on P which are compatible with the preorder in a certain technical sense. To get clear about what we mean by ‘compatible’, we start with the case of a unary operation. We say that a unary operation o is **tonic** just in case one of the following two conditions obtains:

for all $p, q \in P$, if $p \sqsubseteq q$, then $o(p) \sqsubseteq o(q)$ (in this case o is said to be **monotonic** or **preorder-preserving**), or

for all $p, q \in P$, if $p \sqsubseteq q$, then $o(q) \sqsubseteq o(p)$ (in this case o is said to be **antitonic** or **preorder-reversing**).

To generalize to n -ary operations for $n > 1$, we require that the operation be “tonic with respect to each argument when all the other arguments are held fixed”. For example, if o is a binary operation, tonicity means that for each $p \in P$, the unary operations f_p defined by $f_p(q) = o(p, q)$ for all $q \in P$ are either all monotonic or all antitonic; and for each $q \in P$, the unary operations g_q defined by $g_q(p) = o(p, q)$ for all $p \in P$ are either all monotonic or all antitonic.

It is not hard to see that tonic operations “preserve equivalence” in the sense that applying such an operation to equivalent arguments yield equivalent results. For example, if o is a binary tonic operation, $p \equiv q$, and $r \equiv s$, then $o(p, r) \equiv o(q, s)$.

Now let $\langle P, \sqsubseteq, \sqcap \rangle$ be a lower presemilattice. As we saw in Chapter 7, the glb operation \sqcap is monotonic in both arguments, and therefore a lower presemilattice is a particularly simple kind of preordered algebra.

4 Interpreting Implication

We turn now to the operation \rightarrow that will be used to interpret the PL connective \rightarrow (or its natural-language counterparts *implies* and *if... then*).⁴ The basic generalizations about what follows from what, as far as \rightarrow is concerned, are that, for any ϕ , ψ , and ξ ,

ψ follows from $(\phi \rightarrow \psi) \wedge \phi$;

if ψ follows from $(\xi \wedge \phi)$, then $(\phi \rightarrow \psi)$ follows from ξ .

⁴Note that we fail to distinguish notationally between the implication connective and the rpc operation that interprets it, relying on context to disambiguate.

Again, in the case of natural language, with *implies* or *if . . . then* in place of \rightarrow , we can view these as empirical generalizations about how native speakers of English judge the validity of certain kinds of simple arguments. In the setting of standard PL, the first generalization is called **implication elimination** or **modus ponens**, and the second is called **implication introduction** or **Curry's Law**.

The most straightforward way to get our algebraic semantics to make the right predictions about how implication works is to require that \rightarrow have the following properties, for all $p, q, r \in \mathbf{P}$:

$$(p \rightarrow q) \sqcap p \sqsubseteq q; \text{ and}$$

$$\text{if } r \sqcap p \sqsubseteq q, \text{ then } r \sqsubseteq p \rightarrow q.$$

A binary operation on a lower presemilattice that has these properties is called a **relative pseudocomplement** or **rpc** operation.

More generally, if P is a lower presemilattice with preorder \sqsubseteq and glb operation \sqcap , and $p, q, r \in P$, then r is called a **relative pseudocomplement of p relative to q** iff r is a greatest member of the set $\{x \in P \mid x \sqcap p \sqsubseteq q\}$. Clearly all relative pseudocomplements of p with respect to q are equivalent, and there can be at most one of them if \sqsubseteq is antisymmetric.

The following properties of rpc operations are easy to prove, for all p, q , and r :

Un-Curry: if $r \sqsubseteq p \rightarrow q$, then $r \sqcap p \sqsubseteq q$

Antitonicity on First Argument: if $p \sqsubseteq q$ then $(q \rightarrow r) \sqsubseteq (p \rightarrow r)$

Monotonicity on Second Argument: if $p \sqsubseteq q$, then $(r \rightarrow p) \sqsubseteq (r \rightarrow q)$.

Finally, putting the pieces together, a **heyting presemilattice (HPS)** is a set P together with a preorder \sqsubseteq , a top \top , a glb operation \sqcap , and an rpc operation \rightarrow . HPS's provide preordered-algebra interpretations for the fragment of PL, called **positive intuitionistic propositional logic (PIPL)**, whose only connectives are \top , \wedge , and \rightarrow . To be more explicit, an **HPS interpretation** of PIPL is an HPS $\langle P, \sqsubseteq, \sqcap, \rightarrow, \top \rangle$ together with a function **sem** from PIPL formulas to P such that, for all PIPL formulas ϕ and ψ ,

$$\text{sem}(\phi \wedge \psi) = \text{sem}(\phi) \sqcap \text{sem}(\psi)$$

$$\text{sem}(\phi \rightarrow \psi) = \text{sem}(\phi) \rightarrow \text{sem}(\psi)$$

$$\text{sem}(T) = \top.$$

We will return to HPS interpretations, and connect them with a formalized notion of proof, in the following chapter.

In addition to the facts about \sqcap and \rightarrow already noted, some noteworthy facts about heyting presemilattices are the following:

$$p \sqcap \top \equiv p$$

$$p \sqsubseteq q \text{ iff } p \rightarrow q \equiv \top$$

$$p \rightarrow p \equiv \top$$

$$p \sqcap (p \rightarrow q) \equiv p \sqcap q$$

$$(p \rightarrow q) \sqcap q \equiv q$$

$$p \rightarrow (q \sqcap r) \equiv (p \rightarrow q) \sqcap (p \rightarrow r)$$

$$p \rightarrow q \sqcap (q \rightarrow r) \sqsubseteq p \rightarrow r.$$

The proofs of these facts are left as exercises.

5 Generalizing Entailment

In a preorder $\langle P, \sqsubseteq \rangle$, if $q \in P$ and $S \subseteq P$, we say $S \sqsubseteq q$ iff $p \sqsubseteq q$ for every lower bound of S . (So if S has a glb s , $S \sqsubseteq q$ iff $s \sqsubseteq q$.) Notice that the new usage of the symbol ' \sqsubseteq ' conflicts with the previous one, since it denotes not a binary relation on P but rather a relation, called **(generalized) entailment**, between $\wp(P)$ and P . To resolve the conflict, we now regard the notation $p \sqsubseteq q$ (where $p \in P$) as an abbreviation for $\{p\} \sqsubseteq q$. Additionally, we abbreviate $\emptyset \sqsubseteq q$ to $\sqsubseteq q$, and $\{p_1, \dots, p_n\} \sqsubseteq p$ to $p_1, \dots, p_n \sqsubseteq p$.

Some simple facts about entailment (in this generalized sense) are the following:

- a. $\sqsubseteq q$ iff q is a top
- b. if P has a glb operation \sqcap , then $p, q \sqsubseteq r$ iff $p \sqcap q \sqsubseteq r$; and more generally

$$p_1, \dots, p_n \sqsubseteq q \text{ iff } p \sqsubseteq q \text{ for some (and therefore any) glb } p \text{ of } \{p_1, \dots, p_n\}.$$
- c. If additionally P has an rpc operation \rightarrow , then $r \sqsubseteq p \rightarrow q$ iff $r, p \sqsubseteq q$ iff $r \sqcap p \sqsubseteq q$.