Gentzen-Sequent-Style Natural Deduction for PIPL

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These slides are available at:

http://www.ling.ohio-state.edu/∼plummer/ling681

(1) **Deduction**

a. It seems that natural language users are able to reason about the correctness of arguments (whether a certain sentencee must be true if certain other sentences are assumed to be true) based solely on the *forms* of the sentences in question, independently of contingent fact (what happens to be true).

b. We call such reasoning **deduction**.

c. *Proof theories* first arose as as idealizations of NL deduction, much as logical languages first arose as idealizations of NL itself.

d. In the kind of proof theory called *(Gentzen-sequent-style) natural deduction* we model deduction by a system of *axioms* and *rules* that recursively defines what formulas are deducible from what other formulas.

(2) **Provability**

a. We define another relation (besides semantic consequence) between contexts and formulas called **provability** (also called **derivability** or **deducibility**), denoted by $\vdash$.

b. Unlike semantic consequence, provability is defined in purely syntactic terms and makes no reference to HPS (or any other) interpretations.

c. But it will turn out that semantic consequence and provability are the same relation!

d. In other words: whether a given PIPL formula is a semantic consequence of a given finite set of formulas can be determined by purely syntactic means.

(3) **Sequents**

a. A **sequent** is a metalanguage assertion of the form $\Gamma \vdash \psi$: it asserts that the formula ψ is provable from the context Γ.

b. In a sequent, Γ is called the **context**, $\vdash$ is called the **turnstile**, and ψ is called the **succedent**.

c. The members of the context are called the **hypotheses** or **assumptions**.

d. In sequents, the same abbreviatory conventions for contexts are employed as in assertions of semantic consequence (see item (6) on the handout "Generalized Entailment and Semantic Consequence.").

e. We use Σ (possibly subscripted) as a metavariable over sequents.

(4) **The Form of a Natural Deduction (ND) Proof Theory**

a. The theory recursively defines the provability relation

b. The base clauses of the recursive definition are certain sequents called **axioms**.

c. The recursion clauses of the recursive definition are certain metalanguage conditional assertions called **(inference) rules**.

d. In each rule, the antecedent of the conditional is a (metalanguage) conjunction of sequents, and the consequent is a sequent:

if Σ_1 and ... and Σ_n, then Σ.

(5) **Rules**

a. Rules are conventionally written in the format:

$$\frac{\Sigma_1 \;\ldots\; \Sigma_n}{\Sigma}$$

where n is 1 or 2.

b. In a rule, the Σ_i are called the **premisses**, and Σ is called the **conclusion**.

c. An **axiom** can be thought of as a rule with no premisses:

$$\frac{}{\Sigma}$$

(6) **The Axioms**

TI (Truth Introduction)

$$\frac{}{\vdash T}$$

R (Reflexivity)

$$\frac{}{\phi \vdash \phi}$$

(7) **The Rules for Conjunction**

$\wedge$E (Conjunction Elimination 1)

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}$$

$\wedge$E' (Conjunction Elimination 2)

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}$$

$\wedge$I (Conjunction Introduction)

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

(8) **The Rules for Implication**

→E (Implication Elimination *or* Modus Ponens)

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \phi \rightarrow \psi}{\Gamma \vdash \psi}$$

→I (Implication Introduction *or* Hypothetical Proof)

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}$$

(9) **A Structural Rule**

W (Weakening)

$$\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}$$

(10) **Formal Proofs**

A **(formal) proof** of a sequent Σ is a labelled tree with each node labelled by a sequent, such that the following three conditions hold:

a. The root is labelled by Σ.

b. Each leaf is labelled by an axiom.

c. For each nonleaf x, the label of x is the conclusion of a rule whose premisses are the labels of x's daughters.

The pair $\langle \Sigma, \psi \rangle$ is in the relation $\vdash$ iff there is a formal proof of the sequent $\Sigma \vdash \psi$.

In this case, we call $\Gamma \vdash \psi$ a **(PIPL)-theorem**. Equivalently, we say ψ is **(PIPL-)provable** (or **deducible**, or **derivable**) **from** Γ, or Γ **proves** (or **deduces**, or **derives**) ψ (**in PIPL**).

(11) **Soundess and Completeness**

a. PIPL deduction is **sound** relative to the class of HPS interpretations, i.e. if $\Sigma \vdash \psi$ then $\Sigma \models \psi$.

b. PIPL deduction is **complete** relative to the class of HPS interpretations, i.e. if $\Sigma \models \psi$ then $\Sigma \vdash \psi$.

c. In short: provability and semantic consequence are the same relation.

(12) **PIPL theorem:** $\phi \vdash \phi \wedge \phi$

Formal proof:

$$\phi \vdash \phi \wedge \phi$$

$$\phi \vdash \phi \qquad \phi \vdash \phi$$

(13) **PIPL theorem:** $\phi \wedge \psi \vdash \psi \wedge \phi$

Formal proof:

$$\phi \wedge \psi \vdash \psi \wedge \phi$$

$$\phi \wedge \psi \vdash \psi \qquad \phi \wedge \psi \vdash \phi$$

$$\phi \wedge \psi \vdash \phi \wedge \psi \qquad \phi \wedge \psi \vdash \phi \wedge \psi$$

(14) **PIPL theorem:** $\phi \vdash (\phi \to \psi) \to \psi$

Formal proof:

$$\phi \vdash (\phi \to \psi) \to \psi$$

$$\phi, \phi \to \psi \vdash \psi$$

$$\phi, \phi \to \psi \vdash \phi \qquad \phi, \phi \to \psi \vdash \phi \to \psi$$

$$\phi \vdash \phi \qquad \phi \to \psi \vdash \phi \to \psi$$