

Gentzen-Sequent-Style Natural Deduction for PIPL

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(1) Deduction

- a. It seems that natural language users are able to reason about the correctness of arguments (whether a certain sentence must be true if certain other sentences are assumed to be true) based solely on the *forms* of the sentences in question, independently of contingent fact (what happens to be true).
- b. We call such reasoning **deduction**.
- c. *Proof theories* first arose as idealizations of NL deduction, much as logical languages first arose as idealizations of NL itself.
- d. In the kind of proof theory called (*Gentzen-sequent-style*) *natural deduction* we model deduction by a system of *axioms* and *rules* that recursively defines what formulas are deducible from what other formulas.

(2) Provability

- a. We define another relation (besides semantic consequence) between contexts and formulas called **provability** (also called **derivability** or **deducibility**), denoted by \vdash .
- b. Unlike semantic consequence, provability is defined in purely syntactic terms and makes no reference to HPS (or any other) interpretations.
- c. But it will turn out that semantic consequence and provability are the same relation!
- d. In other words: whether a given PIPL formula is a semantic consequence of a given finite set of formulas can be determined by purely syntactic means.

(3) **Sequents**

- a. A **sequent** is a metalanguage assertion of the form $\Gamma \vdash \psi$: it asserts that ψ is provable from Γ .
- b. In a sequent, Γ is called the **context**, \vdash is called the **turnstile**, and ψ is called the **succedent**.
- c. The members of the context are called the **hypotheses** or **assumptions**.
- d. In sequents, the same abbreviatory conventions for contexts are employed as in assertions of semantic consequence.
- e. We use Σ (possibly subscripted) as a metavariable over sequents.

(4) **The Form of a Natural Deduction (ND) Proof Theory**

- a. The theory recursively defines the provability relation
- b. The base clauses of the recursive definition are certain sequents called **axioms**.
- c. The recursion clauses of the recursive definition are certain metalanguage conditional assertions called **(inference) rules**.
- d. In each rule, the antecedent of the conditional is a (metalanguage) conjunction of sequents, and the consequent is a sequent: if Σ_1 and \dots and Σ_n , then Σ .

(5) **Rules**

- a. Rules are conventionally written in the format:

$$\frac{\Sigma_1 \dots \Sigma_n}{\Sigma}$$

where n is 1 or 2.

- b. In a rule, the Σ_i are called the **premisses**, and Σ is called the **conclusion**.
- c. An **axiom** can be thought of as a rule with no premisses:

$$\frac{}{\Sigma}$$

(6) **The Axioms**

TI (Truth Introduction)

$$\frac{}{\vdash T}$$

R (Reflexivity)

$$\frac{}{\phi \vdash \phi}$$

(7) **The Rules for Conjunction**

\wedge E (Conjunction Elimination 1)

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}$$

\wedge E' (Conjunction Elimination 2)

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}$$

\wedge I (Conjunction Introduction)

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

(8) **The Rules for Implication**

\rightarrow E (Implication Elimination *or* Modus Ponens)

$$\frac{\Gamma \vdash \phi \quad \Gamma \vdash \phi \rightarrow \psi}{\Gamma \vdash \psi}$$

\rightarrow I (Implication Introduction *or* Hypothetical Proof)

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}$$

(9) **A Structural Rule**

W (Weakening)

$$\frac{\Gamma \vdash \psi}{\Gamma, \phi \vdash \psi}$$

(10) **Formal Proofs**

A **(formal) proof** of a sequent Σ is a labelled tree with each node labelled by a sequent, such that the following three conditions hold:

- a. The root is labelled by Σ .
- b. Each leaf is labelled by an axiom.
- c. For each nonleaf x , the label of x is the conclusion of a rule whose premisses are the labels of x 's daughters.

The pair $\langle \Sigma, \psi \rangle$ is in the relation \vdash iff there is a formal proof of the sequent $\Sigma \vdash \psi$.

In this case, we call $\Gamma \vdash \psi$ a **(PIPL)-theorem**. Equivalently, we say ψ is **(PIPL-)provable** (or **deducible**, or **derivable**) **from** Γ , or Γ **proves** (or **deduces**, or **derives**) ψ (**in PIPL**).

(11) **Soundness and Completeness**

- a. PIPL deduction is **sound** relative to the class of HPS interpretations, i.e. if $\Sigma \vdash \psi$ then $\Sigma \models \psi$.
- b. PIPL deduction is **complete** relative to the class of HPS interpretations, i.e. if $\Sigma \models \psi$ then $\Sigma \vdash \psi$.
- c. Thus provability and semantic consequence are the same relation.

(12) **PIPL theorem:** $\phi \vdash \phi \wedge \phi$

Formal proof:

$$\begin{array}{c} \phi \vdash \phi \wedge \phi \\ \wedge \\ \phi \vdash \phi \quad \phi \vdash \phi \end{array}$$

