

CHAPTER 11: FULL PROPOSITIONAL LOGICS

1 IPL and Heyting Prelattices

By **full** PLs, we mean ones with the complete inventory of standard connectives: those of PIPL (\wedge , \rightarrow , T), as well as \vee , F , and \neg . In this section we consider **intuitionistic PL (IPL)**, which is a *conservative* extension of PIPL, in the sense that the subset of IPL theorems which employ only the PIPL connectives are the same as the PIPL theorems. The following section will be concerned with **classical PL (CPL)**, a nonconservative, but more familiar, extension of (P)IPL.

1.1 Disjunction

Traditionally, the connective \vee is an idealized counterpart of (inclusive) *or*. And just as we modelled \wedge algebraically by a meet operation \sqcap , we model \vee by a join operation \sqcup . This is justified by the facts that for any English sentences S , S' , and S'' , according to the intuitions of native English speakers:

- a. The disjunctive sentence S' *or* S'' follows from S' .
- b. The disjunctive sentence S' *or* S'' follows from S'' .
- c. If S follows from S' and S follows from S'' , then S follows from the disjunctive sentence S' *or* S'' .

The ND introduction rules for \vee are relatively straightforward:

$\vee I$ (Disjunction Introduction 1)

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}$$

$\vee I'$ (Disjunction Introduction 2)

$$\frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}$$

Note that the form of these rules can be obtained from the *elimination* rules for conjunction by (1) replacing \wedge by \vee , and (2) exchanging premisses and conclusions:

$\wedge E$ (Conjunction Elimination 1)

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}$$

$\wedge E'$ (Conjunction Elimination 2)

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}$$

Unfortunately, we cannot obtain an elimination rule for \vee by applying the same trick to the introduction rule for \wedge . That would yield:

$$\frac{\Gamma \vdash \phi \vee \psi}{\Gamma \vdash \phi \quad \Gamma \vdash \psi}$$

which is clearly wrong if we read the space between the two conclusions as (metalanguage) *and*. If instead we try to read the space between multiple conclusions in a rule as (metalanguage) *or*, we get something that looks more promising; in fact it can be proved as a *metatheorem* about IPL. But it is not really an elimination rule; in fact it is not even an inference rule at all because it doesn't tell us which of the two alternative conclusions to infer! The standard elimination rule for \vee is the following:

$\vee E$ (Disjunction Elimination, or Proof by Cases)

$$\frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, \phi \vdash \xi \quad \Gamma, \psi \vdash \xi}{\Gamma \vdash \xi}$$

Now we have a single conclusion with no occurrence of the \vee connective, as desired. However the presence in the rule of the formula ξ in addition to the two disjuncts ϕ and ψ is somewhat disconcerting; Some logicians refer to the conclusion ξ in this rule as 'parasitic'. Note that unlike our other ND inference rules, this rule has *three premisses*: one with the disjunction as the succedent, and two where the disjuncts are used to prove the parasitic formula.

1.2 False

Traditionally, the connective F (false) is an idealized counterpart of a necessarily false sentence. Correspondingly, we model it algebraically by a bottom (\perp), just as T is modelled as a top (\top). Note that \perp bears the same relation to \sqcup as \top bears to \sqcap : while \top and \sqcap are, respectively, nullary and binary glb operations, \perp and \sqcup are, respectively, nullary and binary lub operations.

Just as T has only an introduction rule, F has only an elimination rule:

FE (False Elimination)

$$\frac{\Gamma \vdash F}{\Gamma \vdash \phi}$$

Note that in the special case where Γ is $\{F\}$ the premiss is an axiom, and so we obtain as a theorem (schema) the sequent $F \vdash \phi$, traditionally known as $EF(S)Q$ (ex falso (sequitur) quodlibet).

1.3 Negation

The negative connective \neg corresponds traditionally to the ‘Mathese’ *it is not the case that* or, somewhat less traditionally, to the ‘Wayne’s World’ (postsentential) *not*. Often, \neg is *defined*, by treating $\neg\phi$ as shorthand for $\phi \rightarrow F$. However, we will treat \neg as a distinct connective. (This will make a difference in NL semantics, where we do not want to be forced to treat the sentences *Frege was not Italian* and *if Frege was Italian, then Saussure was Swiss and Saussure was not Swiss* as expressing the same proposition.) Then the elimination and introduction rules are as follows:

$\neg E$ (Negation Elimination)

$$\frac{\Gamma \vdash \neg\phi \quad \Gamma \vdash \phi}{\Gamma \vdash F}$$

$\neg I$ (Negation Introduction or Proof by Contradiction)

$$\frac{\Gamma, \phi \vdash F}{\Gamma \vdash \neg\phi}$$

Here the elimination rule says that any assumptions which prove both a formula and its negation also prove a necessary falsehood. Intuitively, this corresponds to the naive intuition that for any sentence S , the conjunctive sentence S , *and it is not the case that* S is a contradiction. Note that, had we been treating $\neg\phi$ as just an abbreviation of $\phi \rightarrow F$, then this rule schema would have just been the sub-schema of \rightarrow E (modus ponens) consisting of all the instances where the consequent formula in the succedent of the conditional premiss is F .

The introduction rule corresponds to the intuition that it is reasonable to conclude the denial of a sentence S when assuming S enables us to deduce a contradiction. Again, had we been treating $\neg\phi$ as an abbreviation of $\phi \rightarrow F$, then the introduction rule would have just been the sub-schema of \rightarrow I (hypothetical proof) consisting of all the instances where the consequent formula in the succedent of the conclusion is F .

A note on terminology is called for here. The rule we are calling \neg I is widely known by at least three other names: *indirect proof*, *proof by contradiction*, and *reductio ad absurdum*. It is most unfortunate that all three of these names are also often applied to a different rule, one which is not even derivable in IPL, namely:

RAA (Reductio ad Absurdum)

$$\frac{\Gamma, \neg\phi \vdash F}{\Gamma \vdash \phi}$$

More or less arbitrarily, we will reserve the name ‘Proof by Contradiction’ for \neg I, and the name ‘Reductio ad Absurdum’ for the rule immediately above. But the reader is warned that that this usage is far from universal. The reason for the terminological confusion is that once RAA is added to IPL (resulting in CPL), every formula ϕ is provable from its own double negation $\neg\neg\phi$, and so the two rules collapse into each other! And since CPL is the form of PL that most people learn, the difference between the two rules is easily overlooked.

1.4 Heyting Prelattices and IPL

A **heyting prelattice** is a preordered algebra $\langle P, \sqsubseteq, \sqcap, \sqcup, \rightarrow, \sim, \top, \perp \rangle$ where $\langle P, \sqsubseteq, \sqcap, \rightarrow, \top \rangle$ is a heyting presemilattice, \sqcup is a join operation, \perp is a

bottom, and \sim is a **pseudocomplement** operation, i.e. for every $p \in P$, $\sim p$ is equivalent to $p \rightarrow \perp$. (From this, it follows easily that \sim is antitonic.)

As in all prelattices (Chapter 7), the following equivalences hold in a heyting prelattice:

$$\text{(Absorption u.t.e.) } (p \sqcup q) \sqcap q \equiv q \equiv (p \sqcap q) \sqcup q; \text{ and}$$

$$\text{(Semidistributivity) } (p \sqcap q) \sqcup (p \sqcap r) \sqsubseteq p \sqcap (q \sqcup r)$$

In fact, in a heyting prelattice, it can also be shown that the inequality reverse to semidistributivity also holds. so that, in fact, heyting prelattices are distributive u.t.e.:

$$\text{(Distributivity u.t.e.) } (p \sqcap q) \sqcup (p \sqcap r) \equiv p \sqcap (q \sqcup r)$$

Additionally, the following equivalences hold in a heyting prelattice:

$$\text{(Law of Exponentials u.t.e) } (p \sqcup q) \rightarrow r \equiv (p \rightarrow r) \sqcap (q \rightarrow r)$$

$$\text{(Second DeMorgan Law u.t.e) } \sim(p \sqcup q) \equiv (\sim p) \sqcap (\sim q)$$

The first of these is closely related to the arithmetic law of exponentials $r^{p+q} = r^p \cdot r^q$, though the explanation will have to be postponed until we develop some category theory. The second is essentially a special case of the first, given the equivalence of $\sim p$ and $p \rightarrow \perp$. The other DeMorgan law (with \sqcap and \sqcup reversed) does *not* hold in an arbitrary heyting prelattice (but *does* hold in any *boolean* prelattice, to be defined in the following section.

Some other useful facts about heyting prelattices are the following:

- a. if $p \sqsubseteq q$ then $\sim q \sqsubseteq \sim p$
- b. if $p \equiv q$ then $\neg p \equiv \neg q$
- c. $p \sqcap \sim p \equiv \perp$
- d. $p \sqsubseteq \sim \sim p$
- e. if $p \sqsubseteq \perp$, then $\sim p \equiv \top$
- f. $\sim \perp \equiv \top$
- g. $\sim \top \equiv \perp$

Finally, a *heyting prelattice interpretation* for (the language of) full PL is a function from PL formulas to a heyting prelattice $\langle P, \sqsubseteq, \sqcap, \sqcup, \rightarrow, \sim, \top, \perp \rangle$ such that, for all formulas ϕ and ψ :

- a. $\text{sem}(T) = \top$
- b. $\text{sem}(F) = \perp$
- c. $\text{sem}(\phi \wedge \psi) = \text{sem}(\phi) \sqcap \text{sem}(\psi)$
- d. $\text{sem}(\phi \vee \psi) = \text{sem}(\phi) \sqcup \text{sem}(\psi)$
- e. $\text{sem}(\phi \rightarrow \psi) = \text{sem}(\phi) \rightarrow \text{sem}(\psi)$
- f. $\text{sem}(\neg\phi) = \sim \text{sem}(\phi)$

Then our proof theory for IPL (i.e. our proof theory for PIPL together with $\forall\text{I}$, $\forall\text{I}'$, $\forall\text{E}$, FE , $\neg\text{E}$, and $\neg\text{I}$) is sound and complete with respect to the class of heyting prelattice interpretations.

2 CPL and Boolean Prelattices

CPL is just IPL with the addition of the inference rule RAA:

RAA (Reductio ad Absurdum)

$$\frac{\Gamma, \neg\phi \vdash F}{\Gamma \vdash \phi}$$

With this addition, it becomes possible to prove a number of important theorems which are not provable in IPL. Rather than state them here, we instead introduce the class of preordered algebras—boolean prelattices—for which the CPL proof theory is sound and complete, and then state the algebraic counterparts of those theorems. (Then rendering them into logical form is a trivial exercise.)

A **boolean prelattice** is a heyting prelattice in which either of the following equivalent schemas hold:

(Law of the Excluded Middle (EM) u.t.e) $p \sqcup \sim p \equiv \top$

(Law of Double Negation (DN)) $\sim \sim p \sqsubseteq p$

In a boolean prelattice, an rpc is usually called simply a **relative complement**, and a pseudocomplement (rpc with respect to \perp) is called simply a **complement**.

A *boolean prelattice interpretation* for (the language of) full PL is just a heyting prelattice interpretation from PL formulas to a boolean prelattice. That is, it is a smaller class of interpretations than the class of heyting prelattice interpretations, because the interpretation of \neg is required to be a complement operation, not merely a pseudocomplement operation. The CPL proof theory (i.e. IPL proof theory plus RAA) is sound and complete for the class of boolean prelattice interpretations. The fact that CPL has more theorems than IPL goes hand in hand with (1) the fact that CPL has an additional inference rule, and (2) the class of interpretations is smaller.

Obviously, among the theorems of CPL not provable in IPL are the logical counterparts of EM and DN. Also provable in CPL are (the logical counterparts of):

$$\text{(First DeMorgan Law u.t.e.) } \sim(p \sqcap q) \equiv (\sim p) \sqcup (\sim q)$$

$$\text{(Law of Relative Complements u.t.e.) } p \rightarrow q \equiv \sim p \sqcup q$$

And finally, the following fact about boolean prelattices corresponds to a famous theorem of CPL called *Peirce's Law*:

$$\text{(Peirce's Law) } (p \rightarrow q) \rightarrow p \sqsubseteq p$$

Note that the only algebra operation mentioned is \rightarrow . This shows that CPL is not a conservative extension of PIPL. Later we will see that there are weak logics (such as *intuitionistic linear logic*) to which DN can be added *conservatively*. In such logics, Peirce's Law cannot be derived.