Introduction to TLC

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These slides are available at:

http://www.ling.ohio-state.edu/~plummer/ling681
Typed Lambda Calculi (TLCs)

- Developed by Church and Curry starting in early 1930’s
- Can be viewed model-theoretically (Henkin-Montague perspective) or proof-theoretically (Curry-Howard perspective)
- A TLC is specified by giving its types, its terms, and an equivalence relation on the terms.
- There are different kinds of TLCs, depending on what kind of logic its type system is based on.
- Here we focus on positive TLC, based on PIPL.
- Positive TLC underlies higher-order logic (HOL), widely used for formalizing theories of NL meaning
(2) Types of Positive TLC

a. There are some basic types. For concreteness, here we use a set of types motivated by NL semantics (this choice is not relevant for the time being):

Prop (for propositions, meanings of sentences)
Ind (for individuals, meanings of names)
Bool (for booleans or truth values, extensions of propositions)
Ent (for entities, extensions of individuals)

b. T is a type
c. if $A$ and $B$ are types, so is $A \land B$
d. if $A$ and $B$ are types, so is $A \rightarrow B$
(3) **Terms of Positive TLC (1/2)**

Note: we write ‘⊢ a : A’ to mean term a is of type A.

a. There are some *nonlogical constants*. In the typical application to NL semantics, these are interpreted as word meanings, e.g.:

    ⊢ Fido’ : Ind
    ⊢ bark’ : Ind → Prop
    ⊢ bite’ : (Ind ∧ Ind) → Prop
    ⊢ give’ : (Ind ∧ Ind ∧ Ind) → Prop
    ⊢ believe’ : (Ind ∧ Prop) → Prop
    ⊢ bother’ : (Prop ∧ Ind) → Prop
(4) Terms of Positive TLC (2/2)

b. There is a logical constant \( \vdash * : T \). In application to NL semantics, this is interpreted as semantic vacuity.

c. For each type \( A \) there are variables \( \vdash x_i^A : A \ (i \in \omega) \).

d. If \( \vdash a : A \) and \( \vdash b : B \), then \( \vdash (a, b) : A \land B \).

e. If \( \vdash a : A \land B \), then \( \vdash \pi(a) : A \).

f. If \( \vdash a : A \land B \), then \( \vdash \pi'(a) : B \).

g. If \( \vdash f : A \rightarrow B \) and \( \vdash a : A \), then \( \vdash \text{app}(f, a) : B \).

Note: \( \text{app}(f, a) \) is usually abbreviated to \( f(a) \).

h. If \( \vdash x : A \) is a variable and \( \vdash b : B \), then \( \vdash \lambda_x b : A \rightarrow B \).
(5) Positive TLC Term Equivalences (1/3)

Here, $t, a, b, p$, and $f$ are metavariables ranging over terms.

a. Equivalences for the term constructors:
   i. $t \equiv \ast$;
   ii. $\pi(a, b) \equiv a$;
   iii. $\pi'(a, b) \equiv b$; and
   iv. $(\pi(p), \pi'(p)) \equiv p$
(6) Positive TLC Term Equivalences (2/3)

b. Equivalences for \( \lambda \) (‘lambda conversion’)

\[(\alpha) \ \lambda_x b \equiv \lambda_y[y/x]b;\]
\n\[(\beta) \ [\lambda_x b](a) \equiv [a/x]b; \ \text{and}\]
\n\[(\eta) \ \lambda_x[f(x)] \equiv f, \ \text{provided} \ x \ \text{is not free in} \ f.\]

Note: the notation ‘\([a/x]b\)’ means the term resulting from substitution in \(b\) of all free occurrences of \(x : A\) by \(a : A\). This presupposes no free variable occurrences in \(a\) become bound as a result of the substitution.
c. Equivalences of Equational Reasoning

(\rho) \ a \equiv a

(\sigma) \text{ If } a \equiv a', \text{ then } a' \equiv a.

(\tau) \text{ If } a \equiv a' \text{ and } a' \equiv a'', \text{ then } a \equiv a''.

(\xi) \text{ If } b \equiv b', \text{ then } \lambda x b \equiv \lambda x b'.

(\mu) \text{ If } f \equiv f' \text{ and } a \equiv a', \text{ then } f(a) \equiv f'(a').
Positive TLC can be viewed as a proof theory for PIPL:

- Types correspond to formulas.
- Type constructors correspond to logical connectives.
- Constants correspond to axioms (without hypotheses).
- Variables correspond to hypotheses.
- Term constructors correspond to inference rules:
  a. $(\neg, \neg)$ corresponds to $\land$-introduction
  b. $\pi$ and $\pi'$ correspond to $\land$- elimination
  c. app corresponds to $\to$-elimination (modus ponens)
The Curry-Howard Perspective, First Pass (2/2)

- \(\lambda\)-binding corresponds to \(\to\)-introduction (hypothetical proof)
- Terms correspond to proofs.
- Free variables in terms correspond undischarged hypotheses of proofs.
- Terms containing nonlogical constants correspond to proofs from nonlogical axioms.
- Combinators (closed terms without nonlogical constants) correspond to theorems proved with no hypotheses.
(10) The Henkin-Montague Perspective

A (set-theoretic) interpretation $I$ of a positive TLC assigns to each type $A$ a set $I(A)$ and to each constant $\vdash a : A$ a member $I(a)$ of $I(A)$, subject to the following constraints:

a. $I(T)$ is a singleton;

b. $I(A \land B) = I(A) \times I(B)$;

c. $I(A \to B) \subseteq I(A) \Rightarrow I(B)$.

Note: the set inclusion can be proper, as long as there are enough functions to interpret all functional terms.
(11) **Variable Assignments**

A *variable assignment* relative to an interpretation is a function that maps each variable to a member of the set that interprets the variable’s type.
(12) **Extending an Interpretation (1/2)**

Given a variable assignment \(\alpha\) relative to an interpretation \(I\), there is a unique extension of \(I\), denoted by \(I_\alpha\), that assigns interpretations to all terms, such that:

a. For each variable \(x\), \(I_\alpha(x) = \alpha(x)\).

b. For each constant \(a\), \(I_\alpha(a) = I(a)\).

c. If \(\vdash a : A\) and \(\vdash b : B\), then \(I_\alpha((a, b)) = \langle I_\alpha(a), I_\alpha(b)\rangle\).
(13) **Extending an Interpretation (2/2)**

d. If \( \vdash p : A \land B \), then \( I_\alpha(\pi(p)) \) is the first component (\( = \) projection onto \( I(A) \)) of \( I_\alpha(p) \); and \( I_\alpha(\pi'(p)) \) is the second component (\( = \) projection onto \( I(B) \)) of \( I_\alpha(p) \).

e. If \( \vdash f : A \rightarrow B \) and \( \vdash a : A \), then \( I_\alpha(f(a)) = (I_\alpha(f))(I_\alpha(a)) \).

f. If \( \vdash b : B \), then \( I_\alpha(\lambda_{x \in A}b) \) is the function from \( I(A) \) to \( I(B) \) that maps each \( s \in I(A) \) to \( I_\beta(b) \), where \( \beta \) is the variable assignment that coincides with \( \alpha \) except that \( \beta(x) = s \).
Observations about Interpretations

- Two terms $\vdash a : A$ and $\vdash b : B$ of positive TLC are term-equivalent iff $A = B$ and, for any interpretation $I$ and any assignment $\alpha$ relative to $I$, $I_\alpha(a) = I_\alpha(b)$.
- Another way of stating the preceding is to say that term equivalence (viewed as an equational proof system) is sound and complete for the class of interpretations described in (10-12).
- For any term $a$, $I_\alpha(a)$ depends only on the restriction of $\alpha$ to the free variables of $a$.
- In particular, if $a$ is a closed, then $I_\alpha(a)$ is independent of $\alpha$ so we can simply write $I(a)$.
- Thus, an interpretation for the basic types and constants extends uniquely to all types and all closed terms.