Generalized Entailment and Semantic Consequence
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(1) Generalized Entailment

Suppose \( \langle P, \subseteq, \cap, \top \rangle \) is a lower presemilattice with top. (Intuitively, think of the members as propositions, \( \subseteq \) as entailment, \( \cap \) as the interpretation of \( \text{and} \), and \( \top \) as some necessary truth.)

a. We define \textbf{generalized entailment}, also denoted by \( \sqsubseteq \), to be the following binary relation between \( \wp(P) \) and \( P \): for \( S \subseteq P \) and \( p \in P \),

\[
S \sqsubseteq p \iff p \in UB(LB(S))
\]

b. Intuitively: a proposition \( p \) is entailed by a set of propositions \( S \) iff, for every way things might be, if every member of \( S \) is true with things that way, then so is \( p \).

c. Note that if \( S \) is finite, then \( S \sqsubseteq p \iff q \sqsubseteq p \) where \( q \) is any of the (equivalent) glb’s of \( S \). (Note that the ambiguity of the symbol ‘\( \sqsubseteq \)’ is disambiguated depending on whether a member of \( P \) or a subset of \( P \) is on the left-hand side of the inequality.)

(2) Notation for Generalized Entailment

We notate \( S \sqsubseteq p \) as:

a. \( \sqsubseteq p \), if \( S = \emptyset \)

b. \( q \sqsubseteq p \), if \( S = \{q\} \)

Note that this is consistent with the notation for the case where \( \sqsubseteq \) denotes a binary relation on \( P \).

c. \( p_1, \ldots, p_n \sqsubseteq p \), if \( S = \{p_1, \ldots, p_n\} \)

Note that in this case:

i. The order in which the elements of \( S \) are listed on the left-hand side is irrelevant.

ii. Repetitions on the left-hand side are irrelevant.
Simple Observations about Generalized Entailment

a. \( \sqsubseteq p \text{ iff } p \equiv \top \) (i.e. \( p \) is a top)
b. \( p, q \sqsubseteq r \text{ iff } p \sqcap q \sqsubseteq r \) more generally
c. If additionally \( P \) has an rpc operation \( \rightarrow \), then \( r \sqsubseteq p \rightarrow q \)
    \( \text{iff } r, p \sqsubseteq q \text{ iff } r \sqcap p \sqsubseteq q \).

HPS Interpretation (Review)

Let \( \Phi \) be the set of PIPL formulas over a fixed finite set of propositional letters. Then recall that an HPS interpretation of PIPL is a heyting presemilattice (HPS) \( \langle P, \sqsubseteq, \top, \sqcap, \rightarrow \rangle \). together with a function \( \text{sem} \) from \( \Phi \) to \( P \) that satisfies the following conditions, for all formulas \( \phi \) and \( \psi \):

i. \( \text{sem}(T) = \top \)
ii. \( \text{sem}(\phi \land \psi) = \text{sem}(\phi) \sqcap \text{sem}(\psi) \)
iii. \( \text{sem}(\phi \rightarrow \psi) = \text{sem}(\phi) \rightarrow \text{sem}(\psi) \)

Semantic Consequence

a. By a context, we mean a finite set of formulas.
b. We define a binary relation between contexts and formulas called semantic consequence, denoted by \( \vdash \), as follows:
   \( \Gamma \vdash \psi \text{ iff, for every HPS interpretation, } \text{sem}[\Gamma] \sqsubseteq \text{sem}(\psi) \).

Notation in Assertions of Semantic Consequence

a. We use \( \Gamma \) and \( \Delta \) as metavariables over contexts.
b. A singleton context \( \{ \phi \} \) is abbreviated as \( \phi \), so \( \phi \vdash \psi \) abbreviates \( \{ \phi \} \vdash \psi \).
c. The empty context is abbreviated by writing nothing, so \( \vdash \psi \) abbreviates \( \emptyset \vdash \psi \).
d. Contexts with more than one member are abbreviated by eliminating the set braces, so \( \phi_1, \ldots, \phi_n \vdash \psi \) abbreviates \( \{ \phi_1, \ldots, \phi_n \} \vdash \psi \).
e. \( \Gamma, \Delta \) abbreviates \( \Gamma \cup \Delta \).
f. \( \Gamma, \phi \) abbreviates \( \Gamma \cup \{ \phi \} \).
Some Simple Observations about Semantic Consequence

a. \( \models \psi \iff \text{sem}(\psi) \equiv \top \) for every HPS interpretation.

b. \( \phi \models \psi \iff \text{sem}(\phi) \subseteq \text{sem}(\psi) \) for every HPS interpretation.

c. \( \phi_1, \ldots, \phi_n \models \psi \iff \text{sem}(\phi_1), \ldots, \text{sem}(\phi_n) \subseteq \text{sem}(\psi) \) for every HPS interpretation.