Generalized Entailment and Semantic Consequence

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These slides are available at:

http://www.ling.ohio-state.edu/~plummer/ling681
(1) **Generalized Entailment**

Suppose \( \langle P, \sqsubseteq, \sqcap, \top \rangle \) is a lower presemilattice with top. (Intuitively, think of the members as propositions, \( \sqsubseteq \) as entailment, \( \sqcap \) as the interpretation of *and*, and \( \top \) as some necessary truth.)

a. We define **generalized entailment**, also denoted by \( \sqsubseteq \), to be the following binary relation between \( \wp(P) \) and \( P \): for \( S \subseteq P \) and \( p \in P \),

\[
S \sqsubseteq p \text{ iff } p \in \text{UB}(\text{LB}(S))
\]

b. Intuitively: a proposition \( p \) is entailed by a set of propositions \( S \) iff, for every way things might be, if every member of \( S \) is true with things that way, then so is \( p \).

c. Note that if \( S \) is finite, then \( S \sqsubseteq p \) iff \( q \sqsubseteq p \) where \( q \) is any of the (equivalent) glb’s of \( S \). (Note that the ambiguity of the symbol ‘\( \sqsubseteq \)’ is disambiguated depending on whether a member of \( P \) or a subset of \( P \) is on the left-hand side of the inequality.)
(2) **Notation for Generalized Entailment**

We notate $S \sqsubseteq p$ as:

a. $\sqsubseteq p$, if $S = \emptyset$

b. $q \sqsubseteq p$, if $S = \{q\}$

Note that this is consistent with the notation for the case where $\sqsubseteq$ denotes a binary relation on $P$.

c. $p_1, \ldots, p_n \sqsubseteq p$, if $S = \{p_1, \ldots, p_n\}$

Note that in this case:

i. The order in which the elements of $S$ are listed on the left-hand side is irrelevant.

ii. Repetitions on the left-hand side are irrelevant.
(3) Simple Observations about Generalized Entailment

a. $\sqsubseteq p$ iff $p \equiv \top$ (i.e. $p$ is a top)

b. $p, q \sqsubseteq r$ iff $p \sqcap q \sqsubseteq r$ more generally

c. If additionally $P$ has an rpc operation $\rightarrow$, then $r \sqsubseteq p \rightarrow q$ iff $r, p \sqsubseteq q$ iff $r \sqcap p \sqsubseteq q$.  


(4) **HPS Interpretation (Review)**

Let \( \Phi \) be the set of PIPL formulas over a fixed finite set of propositional letters. Then recall that an **HPS interpretation of PIPL** is a heyting presemilattice (HPS) \( \langle P, \sqsubseteq, \top, \sqcap, \rightarrow \rangle \). together with a function \( \text{sem} \) from \( \Phi \) to \( P \) that satisfies the following conditions, for all formulas \( \phi \) and \( \psi \):

i. \( \text{sem}(T) = \top \)

ii. \( \text{sem}(\phi \land \psi) = \text{sem}(\phi) \sqcap \text{sem}(\psi) \)

iii. \( \text{sem}(\phi \rightarrow \psi) = \text{sem}(\phi) \rightarrow \text{sem}(\psi) \)
(5) Semantic Consequence

a. By a **context**, we mean a finite set of formulas.

b. We define a binary relation between contexts and formulas called **semantic consequence**, denoted by $\models$, as follows:

   $$\Gamma \models \psi$$ iff, for every HPS interpretation, $\text{sem}[\Gamma] \subseteq \text{sem}(\psi)$. 
(6) Notation in Assertions of Semantic Consequence

a. We use $\Gamma$ and $\Delta$ as metavariables over contexts.

b. A singleton context $\{\phi\}$ is abbreviated as $\phi$, so $\phi \vdash \psi$ abbreviates $\{\phi\} \vdash \psi$.

c. The empty context is abbreviated by writing nothing, so $\vdash \psi$ abbreviates $\emptyset \vdash \psi$.

d. Contexts with more than one member are abbreviated by eliminating the set braces, so $\phi_1, \ldots, \phi_n \vdash \psi$ abbreviates $\{\phi_1, \ldots, \phi_n\} \vdash \psi$.

e. $\Gamma, \Delta$ abbreviates $\Gamma \cup \Delta$.

f. $\Gamma, \phi$ abbreviates $\Gamma \cup \{\phi\}$.
Some Simple Observations about Semantic Consequence

a. $\models \psi$ iff $\text{sem}(\psi) \equiv \top$ for every HPS interpretation.

b. $\phi \models \psi$ iff $\text{sem}(\phi) \sqsubseteq \text{sem}(\psi)$ for every HPS interpretation.

c. $\phi_1, \ldots, \phi_n \models \psi$ iff $\text{sem}(\phi_1), \ldots, \text{sem}(\phi_n) \sqsubseteq \text{sem}(\psi)$ for every HPS interpretation.