Meet and join operations

Suppose \( \sqsubseteq \) is a preorder on \( A \) A binary operation \( \sqcap (\sqcup) \) on \( A \) is called a **meet (join)** operation provided, for all \( a, b \in A \), \( a \sqcap b \) \( (a \sqcup b) \) is a glb (lub) of \( \{a, b\} \) (not necessarily the only one).

It follows from the definition of glb that, for all \( a, b, c \in A \):

a. (Meet Elimination 1) \( a \sqcap b \sqsubseteq a \);
b. (Meet Elimination 2) \( a \sqcap b \sqsubseteq b \); and
c. (Meet Introduction) if \( c \sqsubseteq a \) and \( c \sqsubseteq b \), then \( c \sqsubseteq a \sqcap b \).

It follows from the definition of lub that, for all \( a, b, c \in A \):

d. (Join Introduction 1) \( a \sqsubseteq a \sqcup b \);
e. (Join Introduction 2) \( b \sqsubseteq a \sqcup b \); and
f. (Join Elimination) if \( a \sqsubseteq c \) and \( b \sqsubseteq c \), then \( a \sqcup b \sqsubseteq c \).

(Pre-)(semi-)lattices

A preordered set equipped with a meet (join) operation is called a **lower (upper) presemilattice**, and one equipped with both is called a **prelattice**.

A presemilattice (prelattice) whose underlying preorder is an order is called a **semilattice (lattice)**.

Facts about \( \sqcap (\sqcup) \) in a lower (upper) presemilattice

Here ‘u.t.e.’ abbreviates ‘up to equivalence’.

a. (Idempotence u.t.e.) \( a \sqcap a \equiv a \);
b. (Commutativity u.t.e.) \( a \sqcap b \equiv b \sqcap a \);
c. (Associativity u.t.e.) \( (a \sqcap b) \sqcap c \equiv a \sqcap (b \sqcap c) \);
d. (Monotonicity on Both Sides) For each $a \in A$, the function that maps each $b \in A$ to $a \sqcap b \ (b \sqcap a)$ is monotonic.

e. (Substitutivity u.t.e.) If $a \equiv c$ and $b \equiv d$ then $a \sqcap b \equiv c \sqcap d$.

f. The preceding assertions remain true if $\sqcap$ is replaced by $\sqcup$.

g. (Interdefinability) $a \sqsubseteq b$ iff $a \sqcap b \equiv a$ (iff $a \sqcup b \equiv b$).

h. If the preorder is an order, then all occurrences of $\equiv$ in the preceding can be replaced by $=$.

(4) Facts about $\sqcap$ and $\sqcup$ in a prelattice

a. (Absorption u.t.e.) $(a \sqcup b) \sqcap b \equiv b \equiv (a \sqcap b) \sqcup b$;

b. (Semidistributivity) $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup c)$.

(5) Distributive prelattices

a. A prelattice is called distributive if the inequality reverse to Semidistributivity holds: $a \sqcap (b \sqcup c) \sqsubseteq (a \sqcap b) \sqcup (a \sqcap c)$. holds.

b. Thus in a distributive prelattice, we have the following (Distributivity u.t.e.): $a \sqcap (b \sqcup c) \equiv (a \sqcap b) \sqcup (a \sqcap c)$.

c. It can be shown that this equivalence holds in a prelattice just in case the dual one (formed by interchanging meets and joins) does.

(6) Relative pseudocomplement (rpc)

Suppose $\langle A, \sqsubseteq, \sqcap \rangle$ is a lower presemilattice.

a. If $a, b, c \in A$, then $c$ is called a relative pseudocomplement (rpc) of $a$ relative to $b$ iff the following two conditions hold:
   i. $c \sqcap a \subseteq b$; and
   ii. for all $d \in A$, if $d \sqcap a \subseteq b$, then $d \subseteq c$.

b. Equivalently, $c$ is an rpc of $a$ relative to $b$ iff it is a greatest member of $\{x \in A \mid x \sqcap a \subseteq b\}$.

c. It follows that all rpc’s of $a$ relative to $b$ are equivalent.
(7) **RPC Operations**

A binary operation \( \to \) on \( A \) is called an **rpc operation** iff, for all \( a, b \in A \), \( a \to b \) is an rpc of \( a \) relative to \( b \).

It follows from the defining conditions (6) for an rpc that:

a. (RPC Elimination) \((a \to b) \cap a \sqsubseteq b\); and

b. (RPC Introduction) if \( c \cap a \sqsubseteq b \), then \( c \sqsubseteq a \to b \).

Other important properties of rpc operations include these:

c. (Converse of RPC Introduction) if \( c \sqsubseteq a \to b \), then \( c \cap a \sqsubseteq b \).

d. (Antitonicity in 1st argument) if \( a \sqsubseteq b \) then \( (b \to c) \sqsubseteq (a \to c) \).

e. (Monotonicity in 2nd argument) if \( a \sqsubseteq b \), then \( (c \to a) \sqsubseteq (c \to b) \).

f. (Substitutivity u.t.e.) If \( a \equiv c \) and \( b \equiv d \) then \( a \to b \equiv c \to d \).

(8) **Heyting (pre-)semilattices**

A lower presemilattice \( \langle A, \sqsubseteq, \cap \rangle \) equipped with a top \( \top \) and an rpc operation \( \to \) is called a **heyting presemilattice**, and a **heyting semilattice** if \( \sqsubseteq \) is an order.

Some facts about heyting presemilattices:

a. \( a \cap \top \equiv a \)

b. \( a \to a \equiv \top \)

c. \( a \cap (a \to b) \equiv a \cap b \)

d. \((a \to b) \cap b \equiv b \)

e. \( a \to (b \cap c) \equiv (a \to b) \cap (a \to c) \)

f. \((a \to b) \cap (b \to c) \sqsubseteq a \to c \).

g. \( a \sqsubseteq b \) iff \( a \to b \equiv \top \).