(1) Definitions

a. A preorder on a set \( A \) is a binary relation \( \sqsubseteq \) (read ‘less than or equivalent to’) on \( A \) which is reflexive and transitive.

b. An antisymmetric preorder is called an order.

c. The equivalence relation \( \equiv \) induced by the preorder is defined by \( a \equiv b \) iff \( a \sqsubseteq b \) and \( b \sqsubseteq a \). (Note that, if \( \sqsubseteq \) is an order, this reduces to the identity relation on \( A \), and correspondingly \( \sqsubseteq \) is read as ‘less than or equal to’.)

(2) Background Assumptions

Until further notice:

a. \( \sqsubseteq \) is a preorder on \( A \)

b. \( \equiv \) is the equivalence relation it induces

c. \( S \subseteq A \)

d. \( a \in A \)

(3) Definitions

a. We call \( a \) an upper (lower) bound of \( S \) iff, for every \( b \in S \), \( b \sqsubseteq a \) \((a \sqsubseteq b)\).

Suppose moreover that \( a \in S \). Then \( a \) is said to be:

b. greatest (least) in \( S \) iff it is an upper (lower) bound of \( S \);

c. a top (bottom) iff it is greatest (least) in \( A \);

d. maximal (minimal) in \( S \) iff, for every \( b \in S \), if \( a \sqsubseteq b \) \((b \sqsubseteq a)\), then \( b \sqsubseteq a \) \((a \sqsubseteq b)\), so that in fact \( a \equiv b \).
(4) **Observations**

a. If \( a \) is greatest (least) in \( S \), then it is maximal (minimal) in \( S \).

b. All greatest (least) members of \( S \) are equivalent.

c. So if \( \sqsubseteq \) is an order, \( S \) has at most one greatest (least) member, and \( A \) has at most one top (bottom).

(5) **Definitions**

Let \( \text{UB}(S) \) (\( \text{LB}(S) \)) be the set of upper (lower) bounds of \( S \). (Note: if \( S = \{ a \} \), then \( \text{UB}(S) \) (\( \text{LB}(S) \)) is usually written \( \uparrow a \) (\( \downarrow a \)), read ‘up of \( a \’) (‘down of \( a \’).)

a. A least member of \( \text{UB}(S) \) is called a **least upper bound** (**lub**) of \( S \).

b. A greatest member of \( \text{LB}(S) \) is called a **greatest lower bound** (**glb**) of \( S \).

(6) **Observations**

a. Any greatest (least) member of \( S \) is a lub (glb) of \( S \).

b. All lubs (glbs) of \( S \) are equivalent.

c. If \( \sqsubseteq \) is an order, then \( S \) has at most one lub (glb).

d. A lub (glb) of \( A \) is the same thing as a top (bottom).

e. A lub (glb) of \( \emptyset \) is the same thing as a bottom (top).

(7) **Binary glbs and lubs**

If \( S = \{ a, b \} \) and \( S \) has a unique glb (lub), it is written \( a \sqcap b \) (\( a \sqcup b \)).

(8) **Facts about \( \sqcap \) and \( \sqcup \) when \( \sqsubseteq \) is an order**

a. (Idempotence) \( a \sqcap a \) exists and equals \( a \).

b. (Commutativity) If \( a \sqcap b \) exists, so does \( b \sqcap a \), and they are equal.

c. (Associativity) If \( (a \sqcap b) \sqcap c \) and \( a \sqcap (b \sqcap c) \) both exist, they are equal.
d. The preceding three assertions remain true if $∩$ is replaced by $\sqcup$.

e. (Interdefinability) $a \sqsubseteq b$ iff $a \cap b$ exists and equals $a$ iff $a \sqcup b$ exists and equals $b$.

f. (Absorbtion)

i. If $(a \cap b) \sqcup b$ exists, it equals $b$.

ii. If $(a \sqcup b) \cap b$ exists, it equals $b$.

(9) Definitions

a. Suppose $A$ and $B$ are preordered by $\sqsubseteq$ and $\leq$ respectively. Then a function $f: A \to B$ is called:

i. monotonic or order-preserving iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a) \leq f(a')$;

ii. antitonic or order-reversing iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a') \leq f(a)$.

b. A monotonic (antitonic) bijection is called a preorder isomorphism (preorder anti-isomorphism) provided its inverse is also monotonic (antitonic).

c. Two preordered sets are said to be preorder-isomorphic provided there is a preorder isomorphism from one to the other.