REVIEW OF PREORDERS

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These slides are available at:

http://www.ling.ohio-state.edu/~plummer/ling681
(1) **Definitions**

a. A **preorder** on a set $A$ is a binary relation $\sqsubseteq$ (read ‘less than or equivalent to’) on $A$ which is reflexive and transitive.

b. An antisymmetric preorder is called an **order**.

c. The equivalence relation $\equiv$ induced by the preorder is defined by $a \equiv b$ iff $a \sqsubseteq b$ and $b \sqsubseteq a$. (Note that, if $\sqsubseteq$ is an order, this reduces to the identity relation on $A$, and correspondingly $\sqsubseteq$ is read as ‘less than or equal to’.)

(2) **Background Assumptions**

Until further notice:

a. $\sqsubseteq$ is a preorder on $A$

b. $\equiv$ is the equivalence relation it induces

c. $S \subseteq A$

d. $a \in A$
(3) **Definitions**

a. We call \( a \) an **upper** (lower) **bound** of \( S \) iff, for every \( b \in S \), \( b \sqsubseteq a \) (\( a \sqsubseteq b \)).

Suppose moreover that \( a \in S \). Then \( a \) is said to be:

b. **greatest** (least) in \( S \) iff it is an upper (lower) bound of \( S \);

c. a **top** (bottom) iff it is greatest (least) in \( A \);

d. **maximal** (minimal) in \( S \) iff, for every \( b \in S \), if \( a \sqsubseteq b \) (\( b \sqsubseteq a \)), then \( b \sqsubseteq a \) (\( a \sqsubseteq b \)), so that in fact \( a \equiv b \).
(4) **Observations**

a. If $a$ is greatest (least) in $S$, then it is maximal (minimal) in $S$.

b. All greatest (least) members of $S$ are equivalent.

c. So if $\subseteq$ is an order, $S$ has at most one greatest (least) member, and $A$ has at most one top (bottom).

(5) **Definitions**

Let $UB(S)$ ($LB(S)$) be the set of upper (lower) bounds of $S$. (Note: if $S = \{a\}$, then $UB(S)$ ($LB(S)$) is usually written $\uparrow a$ ($\downarrow a$), read ‘up of $a$’ (‘down of $a$’)).

a. A least member of $UB(S)$ is called a **least upper bound (lub)** of $S$.

b. A greatest member of $LB(S)$ is called a **greatest lower bound (glb)** of $S$. 
(6) Observations

a. Any greatest (least) member of $S$ is a lub (glb) of $S$.
b. All lubs (glbs) of $S$ are equivalent.
c. If $\subseteq$ is an order, then $S$ has at most one lub (glb).
d. A lub (glb) of $A$ is the same thing as a top (bottom).
e. A lub (glb) of $\emptyset$ is the same thing as a bottom (top).

(7) Binary glbs and lubs

If $S = \{a, b\}$ and $S$ has a unique glb (lub), it is written $a \sqcap b$ ($a \sqcup b$).
(8) **Facts about \( \sqcap \) and \( \sqcup \) when \( \sqsubseteq \) is an order**

a. (Idempotence) \( a \sqcap a \) exists and equals \( a \).

b. (Commutativity) If \( a \sqcap b \) exists, so does \( b \sqcap a \), and they are equal.

c. (Associativity) If \( (a \sqcap b) \sqcap c \) and \( a \sqcap (b \sqcap c) \) both exist, they are equal.

d. The preceding three assertions remain true if \( \sqcap \) is replaced by \( \sqcup \).

e. (Interdefinability) \( a \sqsubseteq b \) iff \( a \sqcap b \) exists and equals \( a \) iff \( a \sqcup b \) exists and equals \( b \).

f. (Absorption)
   
   i. If \( (a \sqcap b) \sqcup b \) exists, it equals \( b \).
   
   ii. If \( (a \sqcup b) \sqcap b \) exists, it equals \( b \).
(9) Definitions

a. Suppose $A$ and $B$ are preordered by $\sqsubseteq$ and $\leq$ respectively. Then a function $f: A \to B$ is called:

i. **monotonic** or order-preserving iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a) \leq f(a')$;
ii. **antitonic** or order-reversing iff, for all $a, a' \in A$, if $a \sqsubseteq a'$, then $f(a') \leq f(a)$.

b. A monotonic (antitonic) bijection is called a **preorder isomorphism** (**preorder anti-isomorphism**) provided its inverse is also monotonic (antitonic).

c. Two preordered sets are said to be **preorder-isomorphic** provided there is a preorder isomorphism from one to the other.