(Pre-)Algebras for Linguistics
3. Trees

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Linguistics 680:
Formal Foundations

Autumn 2010

Review of Chains

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- Recall also that in a chain, a is minimal (maximal) in a subset S iff it is least (greatest) in S.

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Proof sketch: Let T be the set of natural numbers n such that every ordered set of cardinality $n+1$ has a minimal member, and show that T is inductive.

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Proof sketch: This follows from Theorem 1 together with the fact (just reviewed) that in a chain, a member is least (greatest) iff it is minimal (maximal).

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The rest of the proof consists of showing that the function $f \cup \{< k + 1, a >\}$ is an order isomorphism.

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Trees

A **tree** is a finite set A with an order $\sqsubseteq$ and a top $\top$, such that the covering relation $\prec$ is a function with domain $A \setminus \{\top\}$.

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- Distinct nodes with the same mother are called **sisters**.
- A minimal node (i.e. one with no daughters) is called a **terminal** node.
- A node which is the mother of a terminal node is called a **preterminal** node.

Theorem 4

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Proof: Exercise.

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Proof sketch: Use the RT to define a function $h\colon \omega \to A$, with $X = A$, $x = a$, and F the function which maps non-root nodes to their mothers and the root to itself. Now let $Y = \mathsf{ran}(h)$; it is easy to see that Y is a chain, and that $Y \subseteq \uparrow a$.

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Two distinct nodes in a tree have a glb iff they are comparable.

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  - (No-tangling condition) If a, b, c, d are nodes such that $a < b$, $c \prec a$, and $d \prec b$, then $c < d$.
- In an ordered tree, if $a < b$, then a is said to **linearly precede** b.

Theorem 7

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Proof: Exercise.

Theorem 8

In an ordered tree, the set of terminal nodes ordered by $\leq$ is a chain.

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Proof: Exercise.

CFG Review

- Recall that a **CFG** is an ordered quadruple $\langle T, N, D, P \rangle$ where
  - T is a finite set called the **terminals**;
  - N is a finite set called **nonterminals**
  - D is a finite subset of $N \times T$ called the **lexical entries**;
  - P is a finite subset of $N \times N^+$ called the **phrase structure rules** (PSRs).

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- Recall also these notational conventions:
  - '$A \rightarrow t$ ' means $\langle A, t\rangle \in D$.
  - '$A \rightarrow A_0 \ldots A_{n-1}$' means $\langle A, A_0 \ldots A_{n-1}\rangle \in P$.
  - '$A \rightarrow \{s_0, \ldots s_{n-1}\}$' abbreviates $A \rightarrow s_i$ $(i < n)$.

Phrase Structures for a CFG

- A **phrase structure** for a CFG $\mathbf{G} = \langle T, N, D, P \rangle$ is an ordered tree together with a **labelling** function $\mathbf{l}$ from the nodes to $T \cup N$ such that, for each node a,
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  - $\mathbf{l}(a) \in T$ if a is a terminal node, and
  - $\mathbf{l}(a) \in N$ otherwise.
- Given a phrase structure with linearly ordered (as per Theorem 8) set of terminal nodes $a_0, \ldots, a_{n-1}$ with labels $t_0, \ldots, t_{n-1}$ respectively, the string $t_0 \ldots t_{n-1}$ is called the **terminal yield** of the phrase structure.

Weak and Strong Generative Capacity

- A phrase structure tree is **generated** by the CFG $\mathbf{G} = \langle T, N, D, P \rangle$ iff
  - for each preterminal node with label A and (terminal) daughter with label t, $A \to t \in D$; and
  - for each nonterminal nonpreterminal node with label A and linearly ordered (as per Theorem 7) daughters with labels $A_0, \ldots, A_{n-1}$ respectively, $(n > 0)$, $A \to A_0 \; \ldots A_{n-1} \in P$.

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- The **strong generative capacity** of $\mathbf{G}$ is the set of phrase structures that it generates.
- The **weak generative capacity** of $\mathbf{G}$ is the function $\mathbf{wgc} : N \to T^*$ that maps each nonterminal symbol A to the set of T-strings which are terminal yields of phrase structures generated by $\mathbf{G}$ with root label A.