(Pre-)Algebras for Linguistics

4. Residuation

Carl Pollard

Linguistics 680:
Formal Foundations

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Inflationary and Deflationary Operations

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  - Kleene closure operation on languages

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- If $\circ$ is commutative u.t.e., then there is no difference between a left residual operation and a right residual operation, so we speak simply of a **residual operation**.

Residuated Presemigroups

- A **residuated presemigroup** is a tuple $\langle P, \sqsubseteq, \circ, \multimap_l, \multimap_r \rangle$ where $\langle P, \sqsubseteq, \circ \rangle$ is a presemigroup with left and right residual operations $\multimap_l$ and $\multimap_r$.

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- Example: A^* with language concatenation as $\circ$ and the language residuals as the residual operations (see Ch. 6).

Symmetric Residuated Presemigroups

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- These are relevant in **linear logic**, a kind of propositional logic that underlies certain kinds of categorial grammar, such as **abstract categorial grammar** and **λ-grammar**.

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- Example: We can model the propositions preordered by entailment as a heyting presemilattice with:
  - the meaning of *and* as the meet operation
  - the meaning of *if . . . then* (or *implies*) as the rpc operation