(Pre-)Algebras for Linguistics

6. Modelling Worlds

Carl Pollard

Linguistics 680:
Formal Foundations

Autumn 2010
Some Special Kinds of Subsets of Preorders

If \( \langle P, \sqsubseteq \rangle \) is a preorder and \( S \subseteq P \), then \( S \) is called **upper closed** iff, for every \( p \in S \) and every \( q \in P \), if \( p \sqsubseteq q \), then \( q \in S \).
Some Special Kinds of Subsets of Preorders

- If $\langle P, \sqsubseteq \rangle$ is a preorder and $S \subseteq P$, then $S$ is called **upper closed** iff, for every $p \in S$ and every $q \in P$, if $p \sqsubseteq q$, then $q \in S$.

- If $\langle P, \sqsubseteq, \sqcap \rangle$ is a lower presemilattice and $S \subseteq P$, then $S$ is called:
  - **meet-closed** iff, for every $p, q \in S$, $p \sqcap q \in S$.

- A **filter** iff it is both upper-closed and meet-closed.
- A **proper filter** iff it is a filter and $S \neq P$.
- A **maximal filter** iff it is a proper filter but not a proper subset of a proper filter.

Carl Pollard  (Pre-)Algebras for Linguistics
Some Special Kinds of Subsets of Preorders

- If $\langle P, \sqsubseteq \rangle$ is a preorder and $S \subseteq P$, then $S$ is called **upper closed** iff, for every $p \in S$ and every $q \in P$, if $p \sqsubseteq q$, then $q \in S$.

- If $\langle P, \sqsubseteq, \sqcap \rangle$ is a lower presemilattice and $S \subseteq P$, then $S$ is called:
  - **meet-closed** iff, for every $p, q \in S$, $p \sqcap q \in S$
  - a **filter** iff it is both upper-closed and meet-closed
Some Special Kinds of Subsets of Preorders

- If \( \langle P, \sqsubseteq \rangle \) is a preorder and \( S \subseteq P \), then \( S \) is called \textbf{upper closed} iff, for every \( p \in S \) and every \( q \in P \), if \( p \sqsubseteq q \), then \( q \in S \).

- If \( \langle P, \sqsubseteq, \sqcap \rangle \) is a lower presemilattice and \( S \subseteq P \), then \( S \) is called:
  - \textbf{meet-closed} iff, for every \( p, q \in S \), \( p \sqcap q \in S \)
  - a \textbf{filter} iff it is both upper-closed and meet-closed
  - a \textbf{proper filter} iff it is a filter and \( S \neq P \).
Some Special Kinds of Subsets of Preorders

- If $\langle P, \sqsubseteq \rangle$ is a preorder and $S \subseteq P$, then $S$ is called upper closed iff, for every $p \in S$ and every $q \in P$, if $p \sqsubseteq q$, then $q \in S$.
- If $\langle P, \sqsubseteq, \sqcap \rangle$ is a lower presemilattice and $S \subseteq P$, then $S$ is called:
  - **meet-closed** iff, for every $p, q \in S$, $p \sqcap q \in S$
  - a **filter** iff it is both upper-closed and meet-closed
  - a **proper filter** iff it is a filter and $S \neq P$
  - a **maximal filter** iff it is a proper filter but not a proper subset of a proper filter.
A maximal filter of a pre-boolean algebra is called an ultrafilter.
A maximal filter of a pre-boolean algebra is called an **ultrafilter**.

In the pre-boolean algebra of propositions preordered by entailment, we can use ultrafilters as models of ‘possible worlds’ (ways things might be).
Ultrafilters

- A maximal filter of a pre-boolean algebra is called an **ultrafilter**.
- In the pre-boolean algebra of propositions preordered by entailment, we can use ultrafilters as models of ‘possible worlds’ (ways things might be).
- Let’s see why this is so.
Basic Facts about Ultrafilters

- Theorem 1: a filter $F$ in a pre-boolean algebra is an ultrafilter iff, for every $p \in P$, exactly one of $p$ and $\neg p \in F$. 

- Theorem 2 (Stone's Lemma): if $p \not\sqsubset q$, then there is an ultrafilter $U$ with $p \in U$ but $q \not\in U$.

We need the Axiom of Choice to prove Stone's Lemma. Our remaining assumptions about sets are not sufficient.

Stone (1930s) used this lemma to prove what is now called the Stone Representation Theorem: every boolean algebra is isomorphic to a sub-boolean algebra of a powerset algebra.

Part of the proof is this Corollary of Stone's Lemma: $p \sqsubset q$ iff every ultrafilter containing $p$ also contains $q$. 

Carl Pollard | (Pre-)Algebras for Linguistics
Basic Facts about Ultrafilters

- Theorem 1: a filter $F$ in a pre-boolean algebra is an ultrafilter iff, for every $p \in P$, exactly one of $p$ and $\neg p \in F$.

- Theorem 2 (Stone’s Lemma): if $p \not\subseteq q$, then there is an ultrafilter $U$ with $p \in U$ but $q \notin U$.

We need the Axiom of Choice to prove Stone’s Lemma. Our remaining assumptions about sets are not sufficient.

Stone (1930s) used this lemma to prove what is now called the Stone Representation Theorem: every boolean algebra is isomorphic to a sub-boolean algebra of a powerset algebra.

Part of the proof is this Corollary of Stone’s Lemma: $p \subseteq q$ iff every ultrafilter containing $p$ also contains $q$.
Basic Facts about Ultrafilters

- Theorem 1: a filter $F$ in a pre-boolean algebra is an ultrafilter iff, for every $p \in P$, exactly one of $p$ and $\neg p \in F$.
- Theorem 2 (Stone’s Lemma): if $p \nsubseteq q$, then there is an ultrafilter $U$ with $p \in U$ but $q \notin U$.
- We need the Axiom of Choice to prove Stone’s Lemma. Our remaining assumptions about sets are not sufficient.
Basic Facts about Ultrafilters

- Theorem 1: a filter $F$ in a pre-boolean algebra is an ultrafilter iff, for every $p \in P$, exactly one of $p$ and $\neg p \in F$.

- Theorem 2 (Stone’s Lemma): if $p \not\subseteq q$, then there is an ultrafilter $U$ with $p \in U$ but $q \notin U$.

- We need the Axiom of Choice to prove Stone’s Lemma. Our remaining assumptions about sets are not sufficient.

- Stone (1930s) used this lemma to prove what is now called the Stone Representation Theorem: every boolean algebra is isomorphic to a sub-boolean algebra of a powerset algebra.
Basic Facts about Ultrafilters

- Theorem 1: a filter $F$ in a pre-boolean algebra is an ultrafilter iff, for every $p \in P$, exactly one of $p$ and $\neg p \in F$.

- Theorem 2 (Stone’s Lemma): if $p \not\subseteq q$, then there is an ultrafilter $U$ with $p \in U$ but $q \notin U$.

- We need the Axiom of Choice to prove Stone’s Lemma. Our remaining assumptions about sets are not sufficient.

- Stone (1930s) used this lemma to prove what is now called the **Stone Representation Theorem**: every boolean algebra is isomorphic to a sub-boolean algebra of a powerset algebra.

- Part of the proof is this Corollary of Stone’s Lemma: $p \subseteq q$ iff every ultrafilter containing $p$ also contains $q$. 
Modeling Possible Worlds

These theorems justify modelling possible worlds as the ultrafilters of the preboolean algebra of propositions preordered by entailment (discuss).