PROBLEM SET THREE: RELATIONS AND FUNCTIONS

Problem 1

a. Prove that the composition of two bijections is a bijection.

b. Prove that the inverse of a bijection is a bijection.

c. Let \( U \) be a set and \( R \) the binary relation on \( \wp(U) \) such that, for any two subsets of \( U \), \( A \) and \( B \), \( A R B \) iff there is a bijection from \( A \) to \( B \).

Prove that \( R \) is an equivalence relation.

Problem 2

Let \( A \) be a fixed set. In this question “relation” means “binary relation on \( A \).” Prove that:

a. The intersection of two transitive relations is a transitive relation.

b. The intersection of two symmetric relations is a symmetric relation.

c. The intersection of two reflexive relations is a reflexive relation.

d. The intersection of two equivalence relations is an equivalence relation.

Problem 3

Background. For any binary relation \( R \) on a set \( A \), the symmetric interior of \( R \), written \( \text{Sym}(R) \), is defined to be the relation \( R \cap R^{-1} \). For example, if \( R \) is the relation that holds between a pair of people when the first respects the other, then \( \text{Sym}(R) \) is the relation of mutual respect. Another example: if \( R \) is the entailment relation on propositions (the meanings expressed by utterances of declarative sentences), then the symmetric interior is truth-conditional equivalence.

Prove that the symmetric interior of a preorder is an equivalence relation.

Problem 4

Background. If \( \sqsubseteq \) is a preorder, then \( \text{Sym}(\sqsubseteq) \) is called the equivalence relation induced by \( \sqsubseteq \) and written \( \equiv_{\sqsubseteq} \), or just \( \equiv \) if it’s clear from the context which preorder is under discussion. If \( a \equiv b \), then we say \( a \) and \( b \) are tied with respect to the preorder \( \sqsubseteq \).

Also, for any relation \( R \), there is a corresponding asymmetric relation called the asymmetric interior of \( R \), written \( \text{Asym}(R) \) and defined to be
For example, the asymmetric interior of the love relation on people is the unrequited love relation.

In a context where there is a fixed preorder $\sqsubseteq$, $a \sqsubseteq b$ is usually read “$a$ is less than or equivalent to $b$”; if in addition $\sqsubseteq$ is antisymmetric (i.e. an order), then it is read “$a$ is less than or equal to $b$” because the only thing tied with $a$ is $a$ itself.

In a context where there is a fixed preorder $\sqsubseteq$, $\text{Asym}(\sqsubseteq)$ is usually read “strictly less than”. Careful: if $a \text{Asym}(\sqsubseteq) b$, then not only are $a$ and $b$ not equal, but also they are not equivalent.

If $\sqsubseteq$ is a preorder, then we say $c$ is strictly between $a$ and $b$ to mean that $a$ is strictly less than $c$ and $c$ is strictly less than $b$.

Given a preorder $\sqsubseteq$ on a set $A$ and $a, b \in A$, we say $a$ is covered by $b$ if $a$ is strictly less than $b$ and there is nothing strictly between them. The relation consisting of all such pairs $(a, b)$ is called the covering relation induced by $\sqsubseteq$ and written $\prec\sqsubseteq$, or just $\prec$ when no confusion can arise.

a. Prove that $\prec$ is an intransitive relation.
b. Let $\leq$ be the usual order on $\omega$. What is the induced covering relation? [Hint: We encountered it earlier, under another name.]
c. Let $\leq$ be the usual order on the real numbers. What is the induced covering relation?
d. Let $U$ be a set, $\subseteq_U$ the subset inclusion relation on $\varnothing(U)$, and $\prec$ the corresponding covering relation. In simple English, how do you tell by looking at two subsets $A$ and $B$ of $U$ whether $A \prec B$?

Problem 5

Background. For any binary relation $R$ on $A$, the reflexive closure of $R$, written $\text{Ref}(R)$, is defined to be the relation $R \cup \text{id}_A$. Clearly if $R$ is transitive then $\text{Ref}(R)$ is a preorder.

Now suppose $P$ is the set of all people who have ever lived (i.e. a set that we are using to represent the collection of people who have ever lived) and let $D$ be a transitive asymmetric relation on $P$ used to represent the relation that holds between a pair of people if the first is a descendant of the second. Let $\sqsubseteq = \text{Ref}(D)$, and $\prec$ the corresponding covering relation. To keep things simple, assume (counterfactually, of course) that (1) every person has exactly two parents, and (2) any two people with a parent in common have both of their parents in common.
a. In plain English, why did we require that $D$ be transitive and asymmetric? (That is, what facts of life are modelled by imposing these conditions on $D$?)

b. Write a formula (sentence made up of Mathese symbols) expressing the condition (1). [Hint: it is much easier to express this in terms of $<$ than in terms of $D$!]

c. Write a formula expressing the condition (2). [Same hint as immediately above.]

d. Suppose $a$ and $b$ are two people. Write a formula that means that $a$ and $b$ are cousins. (Yes, it is a bit odd that these people’s names are “a” and “b”.) (To eliminate any variation in or unclarity about the meaning of English kin terms, assume that a person’s cousins are the children of his or her parents’ siblings, not counting ones with whom he or she has a parent in common).

Translate the following formulas into plain English, using familiar kinship terms.

e. $a \prec b$

f. $b \prec^{-1} a$

g. $a \prec o \prec b$

h. $a \left(\prec o \prec^{-1}\right) \\setminus \text{id}_P b$

i. $a \left(\prec^{-1} o \prec\right) \\setminus \text{id}_P b$