CHAPTER THREE: RELATIONS AND FUNCTIONS

1 Relations

Intuitively, a relation is the sort of thing that either does or does not hold between certain things, e.g. the love relation holds between Kim and Sandy just in case Kim loves Sandy, and the less-than relation holds between two natural numbers A and B just in case $A < B$. How should we represent relations mathematically if sets are all we have to work with? A simple-minded first pass might be to represent the love relation as the set of all pairs $\{A, B\}$ such that A and B are two people such that A loves B. (Actually, A and B would not be people at all, but rather certain sets that we have chosen as theoretical standins for (representations of) people: remember that the only things in our mathematical workspace are sets!) Unfortunately, this is too simple, since, for example, we are left with no way to represent unrequited love: what if Kim loves Sandy but Sandy does not love Kim?

A more promising approach is to represent love as the set of *ordered pairs* $\langle A, B\rangle$ such that A loves B. Of course nobody is under the illusion that a set of ordered pairs is the answer Cole Porter had in mind when he wrote *What is this Thing Called Love?* It is what a formal semanticist would call the *extension* of the love relation. (The appropriate way to mathematically represent the actual love relation, as opposed to its extension, is a question we will turn to later when we consider how to represent linguistic meaning.) To take a less vexing example, we can consider the relation $\subseteq_U$ of set inclusion restricted to the subsets of a given set U to be the following set of ordered pairs:[1]

$$\subseteq_U =_{\text{def}} \{\langle A, B\rangle \in \wp(U) \times \wp(U) \mid A \subseteq B\}$$

More generally, we now *define* the notion of relation as follows: a **relation** between A and B is a subset of $A \times B$. Equivalently, it is a set of ordered pairs whose first and second components are in A and B respectively. Equivalently, it is a member of $\wp(A \times B)$. In the special case where $A = B$, we speak of a relation **on** A. For example, $\subseteq_A$ is a relation on $\wp(A)$. But note: according to the way we have defined the notion of a relation, there is no $\subseteq$ relation! (Explaining why is left as an exercise.) As a matter

[1]Please note that on the right-hand side of the following definition, we are making use of a commonplace notational convention whereby $\{\langle x, y\rangle \in A \times B \mid \phi\}$ abbreviates $\{z \in A \times B \mid \exists x \exists y(\phi \wedge z = \langle x, y\rangle)\}$.

of notation, we usually write $a \; R \; b$ to mean that the ordered pair $\langle a, b\rangle$ is in the relation R; that is, $a \; R \; b$ is just another way to say $\langle a, b\rangle \in R$.

An important special case arises when $A = B$ and the relation is

$$\mathsf{id}_A =_{\mathrm{def}} \{\langle x, y\rangle \in A \times A \mid x = y\}$$

This relation is called the **identity** relation on A.

If R is a relation from A to B, the **inverse** of R is the relation from B to A defined as follows:

$$R^{-1} =_{\mathrm{def}} \{\langle x, y\rangle \in B \times A \mid y \; R \; x\}$$

For example, suppose $\leq$ is the standard order on the natural numbers (to be defined precisely later); its inverse is the relation $\geq$. And the inverse of the (extension) of the love relation is the is-loved-by relation. It is easy to see that for any set A,

$$\mathsf{id}_A^{-1} = \mathsf{id}_A$$

and that for any relation R,

$$(R^{-1})^{-1} = R$$

.

As we have seen, a relation is defined as a subset of a cartesian product $A \times B$. More precisely, this should have been called a **binary** relation. Likewise, we can define a **ternary relation** among the sets A, B, and C to be a subset of the threefold cartesian product $A \times B \times C$; thus a ternary relation is a set of ordered triples. For $n > 3$, n-fold cartesian products and n-ary relations are defined in the obvious way.

Recall that a **cartesian power** is a cartesian product all of whose factors are the same, e.g. $A^{(3)} = A \times A \times A$; that $A^{(1)} = A$; and that $A^{(0)} = 1$. Correspondingly, a **unary relation** on A is just a subset of A, and a **nullary relation** on A is a subset of 1, i.e. either 1 or 0.

Suppose R is a relation from A to B and S is a relation from B to C. Then the **composition** of S and R is the relation from A to C defined by

$$S \circ R =_{\mathrm{def}} \{\langle x, z\rangle \in A \times C \mid \exists y \in B(x \; R \; y \wedge y \; S \; z)\}$$

It is easy to see that if R is a relation from A to B, then

$$\mathsf{id}_B \circ R = R = R \circ \mathsf{id}_A$$

Suppose R is a relation from A to B. Then the **domain** and **range** of R are defined as follows:

$$\mathsf{dom}(R) =_{\text{def}} \{x \in A \mid \exists y \in B(x\ R\ y)\}$$

and

$$\mathsf{ran}(R) =_{\text{def}} \{y \in B \mid \exists x \in A(x\ R\ y)\}$$

respectively.

2 Functions

A relation F between A and B is called a **(total) function** from A to B provided for every $x \in A$, there exists a unique $y \in B$ such that $x\ F\ y$).
In that case we write $F \colon A \to B$. This is often expressed by saying that F **takes** members of A as **arguments** and **returns** members of B as **values** (or, alternatively, **takes its values** in B). Obviously,

$$\mathsf{dom}(F) = A$$

For each $a \in \mathsf{dom}(F)$, the unique b such that $a\ F\ b$ is called the **value** of F **at** a, written $F(a)$. Equivalently, we say F **maps** a to b, written $F \colon a \mapsto b$.

In formal semantics, linguistic meanings are often represented as functions of certain kinds. For example, it is fairly standard to represent declarative sentence meanings as functions from a set W of "possible worlds" (which themselves are taken to be representations of different possible ways the world might be) to the set 2 (i.e. $\{0, 1\}$); here 1 and 0 are identified, respectively, with the intutitive notions of truth and falsity. Not quite so straightforward is the use of function terminology by syntacticians, for example referring to the subjects and complements of a verb as its "grammatical arguments". If a verb were really a function, then what would its domain and codomain be? In due course we'll look into the motivation for talking about verbs and other linguistic expressions as if they were functions.

Note that for any set A, the identity relation id_A is the function from A to A such that

$$\mathsf{id}_A(a) = a$$

for every $a \in A$. In some linguistic theories, identity functions serve as the meanings of "referentially dependent" expressions such as pronouns and gaps.

It is not hard to see (after some reflection) that a relation R from A to B is a function from A to B iff

$$R \circ R^{-1} \subseteq \mathsf{id}_B$$

and

$$\mathsf{id}_A \subseteq R^{-1} \circ R$$

We note here a confusing though standard bit of terminology. Given a function $F : A \to B$, we often call B the **codomain** of F. What is confusing is that if B is a proper subset of some other set B', then clearly also $F \colon A \to B'$; but then B' must be the codomain of F! Evidently the notion of codomain of a function is not well-defined. Technically, we can clear up this confusion by defining a **(set theoretic) arrow** from A to B to be an ordered triple $f = \langle A, B, F\rangle$, where $F \colon A \to B$. Now we can unambiguously refer to A and B as the domain and codomain of f, respectively; F is called the **graph** of f. The point is that two distinct arrows can have the same domain and the same graph but different codomains. Thus when we speak (loosely) of a function $F \colon A \to B$ having B as its codomain, we are really talking about the arrow $\langle A, B, F\rangle$. Having called attention to this abuse of language, we will persist in it without further comment.

For any sets A and B, the **exponential** from A to B is the set of arrows from A to B. This is written B^A, read "B to the A". An alternative notation is $A \Rightarrow B$, read "A into B". Note for any set A there is a unique function $\Diamond_A \colon \emptyset \to A$ (what is it?) and a unique function $\Box_A \colon A \to 1$ (what is it?).

A relation F between A and B is called a **partial function** from A to B provided there is a subset $A' \subseteq A$ such that F is a (total) function from A' to B.

For $n \geq 0$, an n-ary **(total) operation** on a set A is a function from $A^{(n)}$ to A. So a unary operation on A is just a function from A to itself, and a nullary operation on A is a function from 1 (i.e. $\{0\}$) to A. It is easy to see that there is a one-to-one correspondence between A and A^1, with each $a \in A$ corresponding to the function from 1 to A that maps 0 to a.

Suppose $F \colon A \to B$. Then F is called:

injective, or **one-to-one**, or an **injection**, if it maps distinct members of A to distinct members of B;

surjective, or **onto**, or a **surjection**, if $\mathsf{ran}(F) = B$; and

bijective, or **one-to-one and onto**, or a **bijection**, or a **one-to-one correspondence**, if it is both injective and surjective.

An important special case of injective functions are defined as follows: if $A \subseteq B$, then the function $\mu_{A,B}\colon A \to B$ that maps each member of A to itself is called the **embedding** of A into B. Note that $\mu_{A,B}$ has the same graph as id_A, but possibly a larger codomain.

Also injective are the functions ι_1 and ι_2, called **canonical injections**, from the cofactors A and B of a coproduct $A+B$ into the coproduct, defined by $\iota_1(a) = \langle 0, a\rangle$ and $\iota_2(b) = \langle 1, b\rangle$ for all $a \in A$ and $b \in B$. Standard examples of surjections are the **projections** π_1 and π_2 of a product $A \times B$ onto its factors A and B respectively, defined by $\pi_1(\langle a, b\rangle) = a$ and $\pi_2(\langle a, b\rangle) = b$ for all $a \in A$ and $b \in B$.

Suppose $A \subseteq B$. Then the function $\chi_A\colon B \to 2$ such that, for each $b \in B$, $\chi_A(b) = 1$ iff $b \in A$ is called the **characteristic** function of A (relative to B). It is easy to see that there is a bijection from $\wp(B)$ to $B \Rightarrow 2$ that maps each subset of B to its characteristic function.

Since functions are relations, the definition of composition for relations makes sense when the two relations being composed are functions. Thus if $F\colon A \to B$ and $G\colon B \to C$, then $G \circ F\colon A \to C$, and for every $x \in A$,

$$G \circ F(x) = G(F(x))$$

It is not hard to see that[2]

$$G \circ F = \{\langle x, z\rangle \in A \times C \mid \exists y \in B(y = F(x) \wedge z = G(y))\}$$

.

For example, taking it one faith for the moment that there is a set ω whose members are precisely the natural numbers, and that the familiar (binary) arithemetic operations (addition, multiplication, and exponentiation) have been given satisfactory set-theoretic definitions (we will make this precise in due course), let F and G be the functions from ω to ω such that

$$F(x) = x^2$$

$$G(x) = x + 2$$

for all $x \in \omega$. Then $G \circ F$ is given by

$$G \circ F(x) = x^2 + 2$$

[2]Please note that in the set description on the right-hand side of the following equation, we make use of a commonplace notational convention whereby $\exists y \in B\phi$ abbreviates $\exists y(\phi \wedge y \in B)$.

Suppose once again that $F : A \to B$ and $G : B \to C$, and suppose moreover that $H\colon C \to D$. Then it is not hard to see that

$$H \circ (G \circ F) = (H \circ G) \circ F$$

Since functions are relations, the following hold for any function $F\colon A \to B$:

$$\mathsf{id}_B \circ F = F = F \circ \mathsf{id}_A$$

$$F \circ F^{-1} \subseteq \mathsf{id}_B$$

$$\mathsf{id}_A \subseteq F^{-1} \circ F$$

Additionally, it is easy to see that F is surjective iff

$$F \circ F^{-1} = \mathsf{id}_B$$

and F is injective iff

$$\mathsf{id}_A = F^{-1} \circ F$$

As a special case, if F is a unary operation on A, then

$$\mathsf{id}_A \circ F = F \circ \mathsf{id}_A = F$$

If in addition F is bijective, then the relation F^{-1} is also a unary operation on A, and

$$F \circ F^{-1} = F^{-1} \circ F = \mathsf{id}_A$$

Suppose $F\colon A \to B$, $A' \subseteq A$, and $B' \subseteq B$. Then the **restriction** of F to A' is the function from A' to B given by

$$F \restriction A' = \{\langle u, v\rangle \in F \mid u \in A'\}$$

Note that this is the same function as $F \circ \mu_{A',A}$. The **image** of A' by F is the set

$$F[A'] =_{\text{def}} \{y \in B \mid \exists x \in A'(y = F(x))\}$$

The **preimage** (or **inverse image**) of B' by F is the set

$$F^{-1}[B'] =_{\text{def}} \{x \in A \mid \exists y \in B'(y = F(x))\}$$

This is more simply described as

$$\{x \in A \mid F(x) \in B'\}$$

3 Special Kinds of Binary Relations

3.1 Properties of Relations

Here we collect some definitions for future reference. Throughout we assume R is a binary relation on A.

Distinct $a, b \in A$, are (R-)**comparable** if either $a\ R\ b$ or $b\ R\ a$; otherwise, they are **incomparable**. R is **connex** iff a and b are comparable for all distinct $a, b \in A$.

R is **reflexive** if $a\ R\ a$ for all $a \in A$ (i.e. $\mathrm{id}_A \subseteq R$). R is **irreflexive** if $a \not R\ a$ for all $a \in A$ (i.e. $\mathrm{id}_A \cap R = \emptyset$).

R is **symmetric** if $a\ R\ b$ implies $b\ R\ a$ for all $a, b \in A$ (i.e. $R = R^{-1}$). R is **asymmetric** if $a\ R\ b$ implies $b \not R\ a$ for all $a, b \in A$ (i.e. $R \cap R^{-1} = \emptyset$). R is **antisymmetric** if $a\ R\ b$ and $b\ R\ a$ imply $a = b$ for all $a, b \in A$ (i.e. $R \cap R^{-1} \subseteq \mathrm{id}_A$). Thus asymmetry is a special case of antisymmetry; more specifically, a relation is asymmetric iff it is both antisymmetric and irreflexive.

A relation R is **transitive** if $a\ R\ b$ and $b\ R\ c$ imply $a\ R\ c$ for all $a, b, c \in A$ (i.e. $R \circ R \subseteq R$). R is **intransitive** if $a\ R\ b$ and $b\ R\ c$ imply $a \not R\ c$ for all $a, b, c \in A$ (i.e. $(R \circ R) \cap R = \emptyset$).

3.2 Orders and Preorders

A **preorder** is a reflexive transitive relation; and an **order** is an antisymmetric preorder. On of the most useful orders overall is the subset relation $\subseteq_A$ on $\wp(A)$. In linguistic applications, as we will see later on, one of the most widely used orders is the *dominance* order on the nodes of a *tree*, used in many syntactic theories to represent the (putative) constituent structure of a linguistic expression. (But not in all syntactic theories; for example, in the family of syntactic theories known as *categorial grammar*, the notion of constituent plays little or no role.)

In many approaches to formal semantics of natural languages, the representations of declarative sentence meanings (usually called *propositions*) are preordered by a relation called *entailment*. Without getting technical at this point, if p and p' are the propositions expressed by two natural-language sentence utterances S and S', p entails p' just in case, no matter what the world is like, if S is true with the world that way, then so is S'. In order to have a formal theory of this, we will have to have a way of set-theoretically representing sentence utterances, propositions, and possible ways the world might be. Considerable care is needed here, since one and the same sentence can express different propositions depending on the context of utterance, and utterances of different sentences can express the same proposition. A controversial issue here is whether or not the entailment relation is antisymmetric. In other words: if two sentences always agree in truth value no matter what the world is like, then must they express the same proposition? We will take up these and related issues in due course.

Let R be a preorder on A, $S \subseteq A$, and $a \in S$. Then a is **maximal** in S if $a\ R\ b$ implies $b\ R\ A$ for every $b \in S$; a is **minimal** in S if $b\ R\ a$ implies $a\ R\ b$ for every $b \in S$. a is **greatest** in S if $b\ R\ a$ for every $b \in S$; a is **least** in S if $a\ R\ b$ for every $b \in S$; and a is a **top** (respectively, **bottom**) if it is greatest (respectively, least) in A. If there is a unique top, it is written $\top_R$. If there is a unique bottom, it is written $\bot_R$. Clearly, if S has any greatest (least) elements, then they (and only they) are maximal (minimal) elements of S.

When the preorder in question is being used to represent the entailment relation on the propositions in a model-theoretic semantics of a natural language, then those propositions (if any) which are true independently of how things are (such propositions are called *necessary truths*) must be tops; and those propositions (if any) which are false no matter how things are (such propositions are called *necessary falsehoods*) must be bottoms. (Why?) Propositions which are neither necessary truths nor necessary falsehoods are called *contingent*; their truth or falsity depends on how things are.

Now suppose the preorder R is also antisymmetric (i.e. it is an order). Then S can have at most one greatest (or least) member; in particular, there can be at most one top (or bottom). If a is greatest (or least) in S, then it is the unique maximal (or minimal) element of a.

But it is possible (even if R is antisymmetric) for a to be the unique maximal (or minimal) element of S without being the greatest (or least) element in S. For that matter, S can have more than one maximal (or minimal) element without any of them being greatest (or least). It's an instructive exercise to try to verify the foregoing assertions by constructing suitable examples.

In a connex preorder, for a to be maximal (minimal) in S is the same thing as for a to be greatest (least) in S. A connex order is called a **chain**, a **total order**, or a **linear order**. A chain is called a **well-ordering** provided every non-empty subset of A has a least element. The standard example of a well-ordering is the standard ($\leq$) order on the natural numbers.

For linguists, the most familiar chains are the *linear precedence (LP)* orders that arise in the representation of the consituent structure (within linguistic theories that countenance such things) of a linguistic expression by an ordered tree, namely (1) the LP order on the daughters (immediate consituents) of a nonterminal node, and (2) the LP order on the preterminals. We will take a close look at the use of tree representations in syntax in Chapters Seven and Eight.

3.3 Equivalence Relations

An **equivalence** relation is a symmetric preorder. If R is an equivalence relation, then for each $a \in A$ the (R-)**equivalence class** of a is

$$[a]_R =_{\text{def}} \{b \in A \mid a \; R \; b\}$$

Usually the subscript is dropped when it is clear from context which equivalence relation is in question. The members of an equivalence class are called its **representatives**. Note that the set of equivalence classes (written A/R and called the **quotient** of A by R) is a **partition** of A, i.e. it is pairwise disjoint and its union is A. It's easy to see that the function from A to A/R that maps each member of A to its equivalence class is a surjection. More generally, for any function $F \colon A \to B$, there is an equivalence relation $\equiv_F$, with two members of A being equivalent just in case F maps them to the same member of B.

If R is a preorder on A, the relation $\equiv_R$ defined by $a \equiv_R b$ iff both $a \; R \; b$ and $b \; R \; a$ is easily seen to be an equivalence relation. In the special case where R is the entailment relation between propositions in a semantic theory, this equivalence relation is called *truth-conditional equivalence*. Thus truth-conditionally equivalent propositions are true under exactly the same conditions. In semantic theories where entailment is taken to be antisymmetric, truth-conditionally equivalent propositions are identical.