Chapter 7

INDEXED AND INTERNAL CATEGORIES

7.1 Indexed Categories

In this section we introduce the basic notions of the Theory of Indexed Categories. In order to improve readability, the following exposition is an (over)simplification of the usual, and more general, approach. In particular, many of the concepts we define up to equality can be defined up to a fixed collection of *canonical* isomorphisms. In this case, the indexed notions introduced in the theory are required to satisfy a suitable set of coherence conditions, which play a quite marginal role, but conversely can easily puzzle the reader who is approaching the Theory of Indexed Categories for the first time. The reader who is interested in more notions in this branch of Category Theory should consult the References.

7.1.1 Definition *Let* ***CAT*** *be the (meta)category of all categories, and S be a category. An* ***S-indexed category*** *is a functor* $A: S^{op} \to \mathbf{CAT}$.

More explicitly, an S-indexed category A is defined by the following data:

i. for every object s of S, a category A(s), called the category of s-indexed families of objects of A;

ii. for every morphism f: s→s' of S, a functor A(f): F(s')→F(s), called the substitution functor determined by f, and frequently denoted as f^*.

Example A simple but important example is the S-indexing S/: $S^{op} \to \mathbf{CAT}$ of S itself. S/ takes every object r of S to the comma category S/r. Remember that the objects of S/s are arrows h: s→r with codomain r. These arrows should be intuitively thought of as families $\{h^{-1}(i)/ i\in r\}$. If f: s→s' is an arrow of S, then f^*: S/s'→S/s is the pulling back functor. Note that pullbacks are usually defined only up to isomorphism, while we are here implicitly supposing a canonical choice. As a matter of fact, the pullback and the associated “functor” are the basic examples of notions profitably defined up to isomorphism, which we mentioned in the introduction.

7.1.2 Definition *Let* $A, B: S^{op} \to \mathbf{CAT}$ *be two S-indexed categories.*

1. The ***product category*** $A\times B: S^{op} \to \mathbf{CAT}$ *is defined by*

$$A\times B(s) = A(s)\times B(s)$$
$$A\times B(f) = A(f)\times B(f);$$

2. The ***dual category*** $A^{op}: S^{op} \to \mathbf{CAT}$ *is defined by*

$$A^{op}(s) = A(s)^{op}$$

$$A^{op}(f) = A(f)^{op}$$

where $A(f)^{op} : A(s')^{op} \to A(s)^{op}$ *is the defined in the obvious way;*

3. If r *is an object of* S, *the S-indexed category* $\boldsymbol{A^r}$ *is defined by*

$$A^r(s) = A(r\times s)$$

$$A(f) = A(id_r\times f).$$

7.1.3 Definition *Let A, B be two S-indexed categories. An* ***S-indexed functor*** $H: A\to B$ *is a natural transformation from* $A: S^{op}\to\mathbf{CAT}$ *to* $B: S^{op}\to\mathbf{CAT}$.

Thus, an S-indexed functor H: A→B is a collection of functors H(s): A(s)→B(s), for s object of S, such that for any f: s→s' in S, $H(s) \circ A(f) = B(f) \circ H(s')$ $(H(s) \circ f^* = f^* \circ H(s'))$.

Given two indexed functors H: A→B and K: B→C, their composition K ∘ H : A→C is defined component-wise (being the composition of natural transformations), i.e., $(K \circ H)(s) = K(s) \circ H(s)$. The identity id_A: A→A, is the identity natural transformation from A to A.

7.1.4 Definition *Let* $H: A\to B$, $K: A\to B$, *be two S-indexed functors. An* ***S-indexed natural transformation*** $\tau: H\to K$ *consists of a natural transformation* $\tau(s): H(s)\to K(s)$ *for any object* s *of* S *such that, for any* $f: s\to s'$ *in S,*

$$(\dagger) \qquad \tau(s) \circ A(f) = B(f) \circ \tau(s') \qquad (\tau(s) \circ f^* = f^* \circ \tau(s')\ ) .$$

The previous definition is more complex than it seems at first sight. Note that $\tau(s): H(s)\to K(s)$, $\tau(s'): H(s')\to K(s')$ are natural transformations, while $A(f): A(s')\to A(s)$ and $B(f): B(s')\to B(s)$ are functors. We are thus composing natural transformations and functors in the way described at the end of section 3.2. $\tau(s) \circ A(f)$ and $B(f) \circ \tau(s')$ are natural transformations of the following type:

$$\tau(s) \circ A(f) : H(s) \circ A(f) \to K(s) \circ A(f)$$

$$B(f) \circ \tau(s') : B(f) \circ H(s') \to B(f) \circ K(s').$$

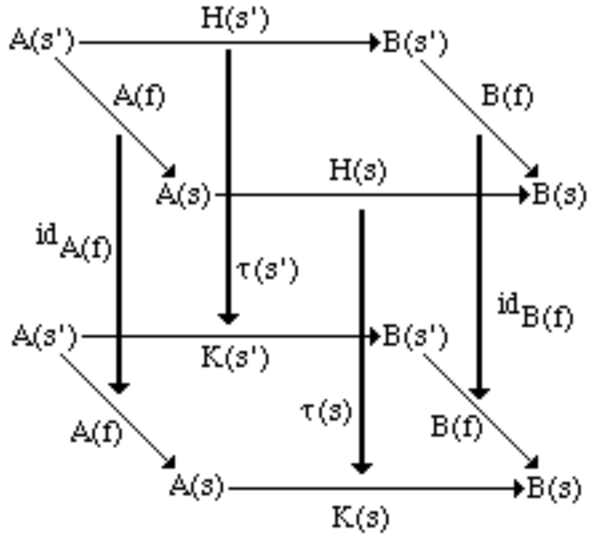
But, according to the definition of S-indexed functors, for any f: s→s', one has $H(s) \circ A(f) = B(f) \circ H(s')$ and $K(s) \circ A(f) = B(f) \circ K(s')$, thus equation (†) is well typed.

Spelling out the composition of natural transformations and functors in (†), we have for any f: s→s' in S and any object a in A(s'),

$$\tau(s)_{A(f)(a)} = B(f)(\tau(s')_a)$$

where the previous equation holds in the category B(s).

The previous situation can be summarized in the following diagram:



(Vertical) composition of S-indexed natural transformations is defined componentwise, that is, given H, K, L : A→B, τ: H→K and ρ: K→L, ρ ∘ τ: H→L is given by $(\rho \circ \tau)(s) = \rho(s) \circ \tau(s)$. This is a good definition since, for any f: s→s' in S and any object a in A(s'),

$$
\begin{aligned}
(\rho \circ \tau)(s)_{A(f)(a)} &= (\rho(s) \circ \tau(s))_{A(f)(a)} \\
&= \rho(s)_{A(f)(a)} \circ \tau(s)_{A(f)(a)} \\
&= B(f)(\rho(s')_a) \circ B(f)(\tau(s')_a) \\
&= B(f)(\rho(s')_a \circ \tau(s')_a) \\
&= B(f)((\rho \circ \tau)(s')_a).
\end{aligned}
$$

7.1.5 Definition *Let A, B be S-indexed categories, H: A→B, K: B→A be S-indexed functors, and* $\eta: id_A \to K \circ H$, $\varepsilon: H \circ K \to id_B$ *be S-indexed natural transformations.* <H, K, η, ε> : A→B *is an* ***S-indexed adjunction*** *if and only if*

$$(K\varepsilon) \circ (\eta K) = id_K$$
$$(\varepsilon H) \circ (H\eta) = id_H .$$

The notion of indexed adjunction is the obvious generalization of the usual notion of adjunction. In particular it is easy to check that for any object s of S, <H(s), K(s), η(s), ε(s)> : A(s)→B(s) is an adjunction in the usual sense.

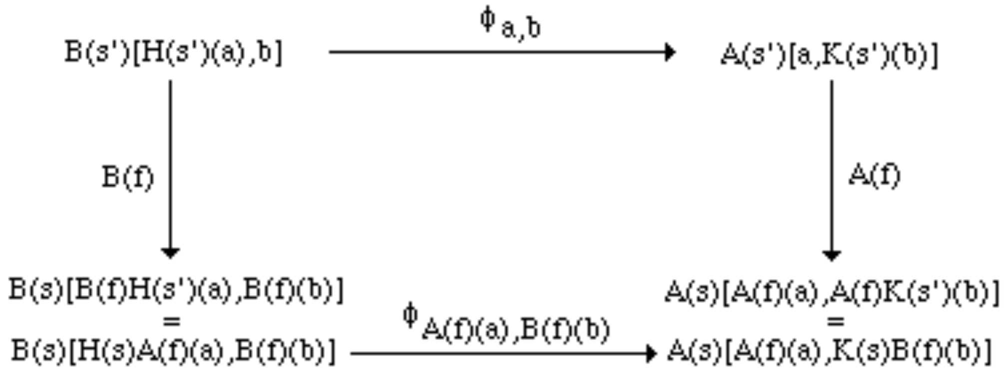
The main problem with the definition of adjunction as a quadruple <H, K, η, ε> : A→B is in its generalization of the case with parameters (remember that the definition of exponents requires an adjunction of this kind). As a triple, an indexed adjunction can be defined in the following, somewhat informal, way:

7.1.6 Definition *Let A, B be S-indexed categories, and H: A→B, K: B→A be S-indexed functors.* *<H, K, φ> : A→B is an* ***S-indexed adjunction*** *if and only if, for every f: s→s' in S,* *i.* $\langle H(s), K(s), \phi(s)\rangle : A(s)\to B(s)$ *is an adjunction*

ii. $\phi(s) \circ B(f) = A(f) \circ \phi(s')$ $\quad(\phi(s) \circ f^* = f^* \circ \phi(s'))$

Equation ii expresses the naturality of the isomorphism ϕ with respect to the index s. Spelling out the composition in ii, we can say that for any f: s→s', a in A(s'), b in B(s'), and g: H(s')(a)→b in B(s'),

$$\phi(s)_{A(f)(a),B(f)(b)}\,(B(f)(g)) = A(f)\,(\phi(s')_{a,b}(g))$$



Suppose we have an adjunction $\langle H, K, \eta, \varepsilon\rangle : A\to B$. Then we obtain ϕ in definition 7.1.6 by letting, for any a in A(s), b in B(s), and g: H(s)(a)→b in B(s),

$$\phi(s)_{a,b}(g) = \varepsilon(s)_b \circ H(s)(g)$$

As we know from chapter 5, for any s in S, $\phi(s)_{a,b}$: B(s)[H(s)(a),b]→A(s)[a,K(s)(b)] is an isomorphism. We now prove that the previous definition of $\phi(s)$ satisfies equation ii in definition 7.1.6. For any f: s→s', a in A(s'), b in B(s'), and g: H(s')(a)→b in B(s'), we have

$A(f)\,(\phi(s')_{a,b}(g))$	$= A(f)\,(\varepsilon(s')_b \circ H(s)(g))$	by def. of $\phi(s')$
	$= A(f)\,(\varepsilon(s')_b) \circ A(f)\,(H(s)(g))$	since A(f) is a functor
	$= \varepsilon(s)_{B(f)(b)} \circ H(s)(B(f)(g))$	by naturality of ε and H
	$= \phi(s)_{A(f)(a),B(f)(b)}\,(B(f)(g))$	by def. of $\phi(s)$

Conversely, given an adjunction $\langle H, K, \phi\rangle : A\to B$, we obviously obtain η, ε by the following:

$$\eta(s)_a = \phi(s)_{a,H(s)(a)}(id_{H(s)(a)}) : a\to K(s)H(s)a$$

$$\varepsilon(s)_b = \phi(s)^{-1}{}_{K(s)(b),b}(id_{K(s)(b)}) : H(s)K(s)b\to b.$$

Definition 7.1.6 has a straightforward generalization to the case with parameters.

7.1.7 Definition *Let A, B, D be S-indexed categories, and H: A×D→B, K:* D^{op}*×B→A be S-indexed functors.* $\langle H, K, \phi\rangle : A\to B$ *is an* ***S-indexed adjunction with parameters in D*** *if and only if, for every f: s→s' in S,*

i. $\langle H(s), K(s), \phi(s)\rangle : A(s)\to B(s)$ *is an adjunction with parameters in D(s);*

ii. $\phi(s) \circ B(f) = A(f) \circ \phi(s')$ $\quad(\phi(s) \circ f^* = f^* \circ \phi(s'))$.

7.2 Internal Category Theory

A category C is **small** when the collection Mor_C of its morphisms is a set. Clearly, then, the collection Ob_C of objects of C is also a set. Moreover, there are set-theoretic functions DOM,COD: $Mor_C \to Ob_C$ that specify source and target of every morphism, a function ID: $Ob_C \to Mor_C$ that defines the identity morphism for every object, and a partial function COMP: $Mor_C \times Mor_C \to Mor_C$ for the composition. Given two morphisms f and g, their composition is defined if and only if DOM(f) = COD(g); the domain of COMP is thus the set {(f,g) | DOM(f) = COD(g)}, that is, the pullback of the two functions DOM,COD: $Mor_C \to Ob_C$. All these functions must also satisfy the obvious equations stating the behavior of the identity morphism with respect to composition, the associativity law for compositition, and the rules which specify domain and target for the identity morphism and for the result of a composition. Thus every small category may be completely described *internally* to the category **Set**, which becomes a sort of "universe of discourse." The previous discussion, however, has made very little use of the specific structure of **Set**; we only needed the existence of pullbacks in order to define the correct domain of the function COMP. In this section, we will show that most of the basic definitions of Category Theory, such as category, functor, natural transformation and so on, can be recasted *inside* any category with *all finite limits*. This means that any such a category may be considered a fairly big universe inside which we can carry out constructions with almost the same confidence as we do in **Set**. This branch of Category Theory is known as "internal," since it describes notions of Category Theory by using the categorical language as a metalanguage.

For many fields of mathematics, from Set Theory to Algebra and Geometry, treatments in the language of Category Theory, even of well-known results, have never been worthless since most of the time they created a new, sometimes unexpected, sense of explanation. The same holds for Category Theory itself: in a sense, Internal Category Theory plays with respect to the general theory the same role that Category Theory plays with respect to Set Theory. If a notion of Category Theory cannot be described internally in a simple way, then there is surely something in that notion that is worth spelling out. As we shall see, this is, for example, the case of the hom-functor and, more generally, of every presheaf.

Internal Category Theory allows us to work in different universes than **Set**. This possibility turns out to be very relevant in several cases, and in particular for the application we aim at in chapter 11, where Internal Category Theory will be applied to the study of categorical models for the polymorphic lambda calculus. In that case, the possibility of working in more constructive categories than **Set** turns out to be essential, as it is known that the standard set-theoretic interpretation of the first order typed lambda calculus cannot be extended to a model of the second order typed lambda calculus.

In the following, E will always denote a category with all finite limits. Our first step is to mimic within E the presentation, within **Set**, of a small category. Thus the collections of objects and morphisms will be viewed as objects of E.

Notation We write $X\times_0 Y$ (instead of $X\times_Z Y$) for the pullback of X and Y along morphisms with common target Z; $<,>_0$ will be used as a "pullback pairing" map, that is, given (suitable) $h:W\to X$ and $k:W\to Y$, we have $<h,k>_0:W\to X\times_0 Y$; the pullback projections will be usually (but not always) denoted by the upper case Greek letter Π, indexed with a number or some other symbol.

7.2.1 Definition *$c = (c_0,c_1,DOM,COD,COMP, ID)$ is an* ***internal category*** *of E ($c\in Cat(E)$) iff:*

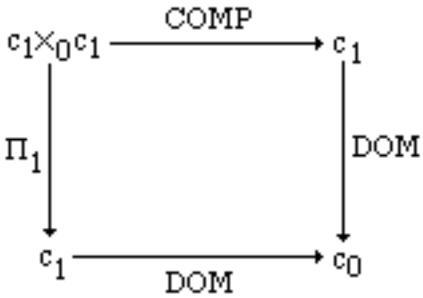
$c_0, c_1\in Ob_E$

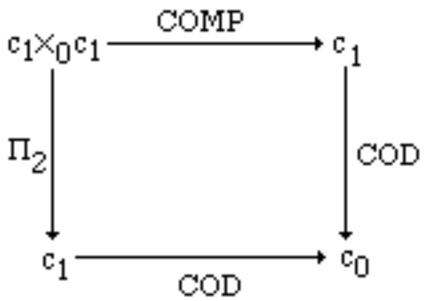
$DOM, COD : c_1 \to c_0$

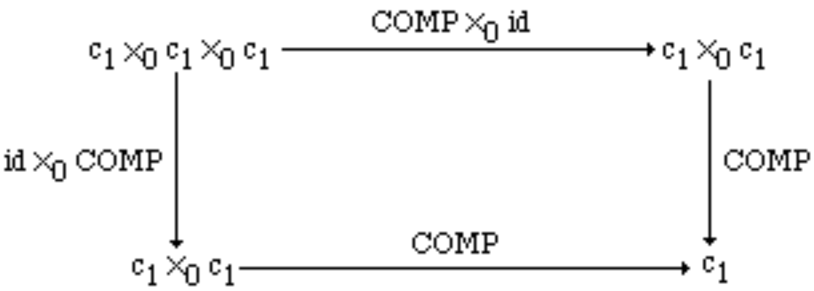
$COMP : c_1 \times_0 c_1 \to c_1$ *where* $c_1 \times_0 c_1$ *is the pullback of* $DOM, COD : c_1\to c_0$

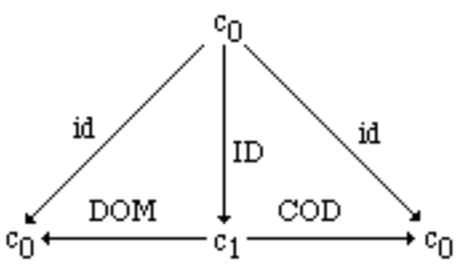
$ID : c_0 \to c_1$

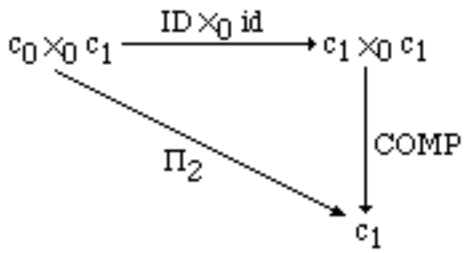
and moreover

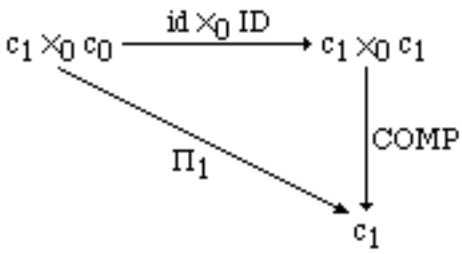












Note that in the diagram expressing the associativity of composition there is an implicit isomorphism between $c_1 \times_0 (c_1 \times_0 c_1)$ and $(c_1 \times_0 c_1) \times_0 c_1$. Indeed,

$$\text{COMP} \circ (\text{COMP} \times_0 \text{id}) : (c_1 \times_0 c_1) \times_0 c_1 \to c_1$$

$$\text{COMP} \circ (\text{id} \times_0 \text{COMP}) : c_1 \times_0 (c_1 \times_0 c_1) \to c_1.$$

In the following, this isomorphism will be always skipped in order to maintain the notation at a simpler level.

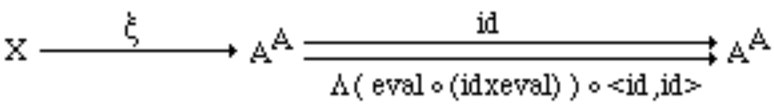
7.2.2 Examples 1. Given an object e in E, the internal **discrete category** associated with e is $(e,e,id_e,id_e,id_e,id_e)$.

2. Let E be a CCC with all finite limits, and let A be an object of E. It is possible to define internally to E a category that plays the role of the category of retractions over A.

Let $m = \Lambda(\, eval \circ (id \times eval)\,) : A^A \times A^A \to A^A$ the internal composition map, that is let $m = \lambda(f,g).g \circ f$. Since E has all finite limits, it has equalizers for every pair of morphisms. Let then (X, ξ) be the equalizer of

$$id : A^A \to A^A$$

$$m \circ \langle id,id \rangle : A^A \to A^A$$



The function $m \circ \langle id,id \rangle : A^A \to A^A$ is $\lambda f.\, f \circ f$; thus the object X represents the subset of A^A of all those functions f such that $f = f \circ f$, i.e., X is an internalization for the set of retractions in A^A. X plays the role of c_0 in the internal category we are defining.

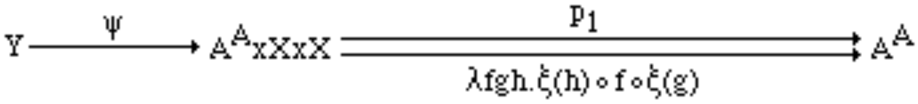
Intuitively, a morphism between two retractions g and h is a triple (f,g,h), where f is a function from A to A such that $f = h \circ f \circ g$.

In order to internalize this definition we use the equalizer (Y,ψ) of

$$p_1 : A^A \times X \times X \to A^A$$

$$m \circ (m \times id) \circ \langle \xi \circ p_3 , p_1, \xi \circ p_2 \rangle : A^A \times X \times X \to A^A$$

Note that $m \circ (m \times id) \circ \langle \xi \circ p_3 , p_1, \xi \circ p_2 \rangle$ is just $\lambda fgh.\, \xi(h) \circ f \circ \xi(g)$.



COD and DOM are obviously defined by the following equations:

$$DOM = p_2 \circ \psi$$

$$COD = p_3 \circ \psi$$

For ID, note first that by definition of ξ, $m \circ < \xi, \xi > = id \circ \xi = \xi$ and, therefore,

$$(\lambda fgh.\ \xi(h) \circ f \circ \xi(g)) \circ < \xi, id, id > = \xi.$$

Thus $< \xi, id, id >: X \to A^A \times X \times X$ equalizes p_1 and $\lambda fgh.\ \xi(h) \circ f \circ \xi(g)$, and $ID: X \to Y$ is the unique arrow such that $\psi \circ ID = < \xi, id, id >$. Note that

$$DOM \circ ID = p_2 \circ \psi \circ ID = p_2 \circ < \xi, id, id > = id$$

$$COD \circ ID = p_3 \circ \psi \circ ID = p_3 \circ < \xi, id, id > = id.$$

Finally, we must define $COMP: Y \times_0 Y \to Y$. The idea is that $(f,g,h) \circ (f',k,g) = (f \circ f', k, h)$. We start defining an arrow $M: Y \times_0 Y \to A^A \times X \times X$ such that $M((f,g,h), (f',k,g)) = (f \circ f', k, h)$; next we prove that M equalizes p_1 and $\lambda fgh.\ \xi(h) \circ f \circ \xi(g)$. Then COMP is the unique arrow from $Y \times_0 Y$ to Y such that $\psi \circ COMP = M$.

3. Given a function $f: U \to C$, consider the C-indexed collection of sets $\{G(c) = f^{-1}(c)\}_{c \in C}$. We can form a small category **C**, which has C as set of objects, and with hom-sets **C**[c,c'] = **Set**[G(c),G(c')]. Composition and identities are inherited from **Set**. The previous construction can be generalized to a generic topos E: given $f: U \to C$ in E, there is an internal category Full(f) that plays the role of the full subcategory generated by the fibers of f. $Full(f)_0$ is C; $Full(f)_1$, together with the map $<DOM,COD>: Full(f)_1 \to C \times C$, is defined as the exponent $p_1{}^*(f)^{p_2{}^*(f)}$ in the slice category $E/C \times C$, where

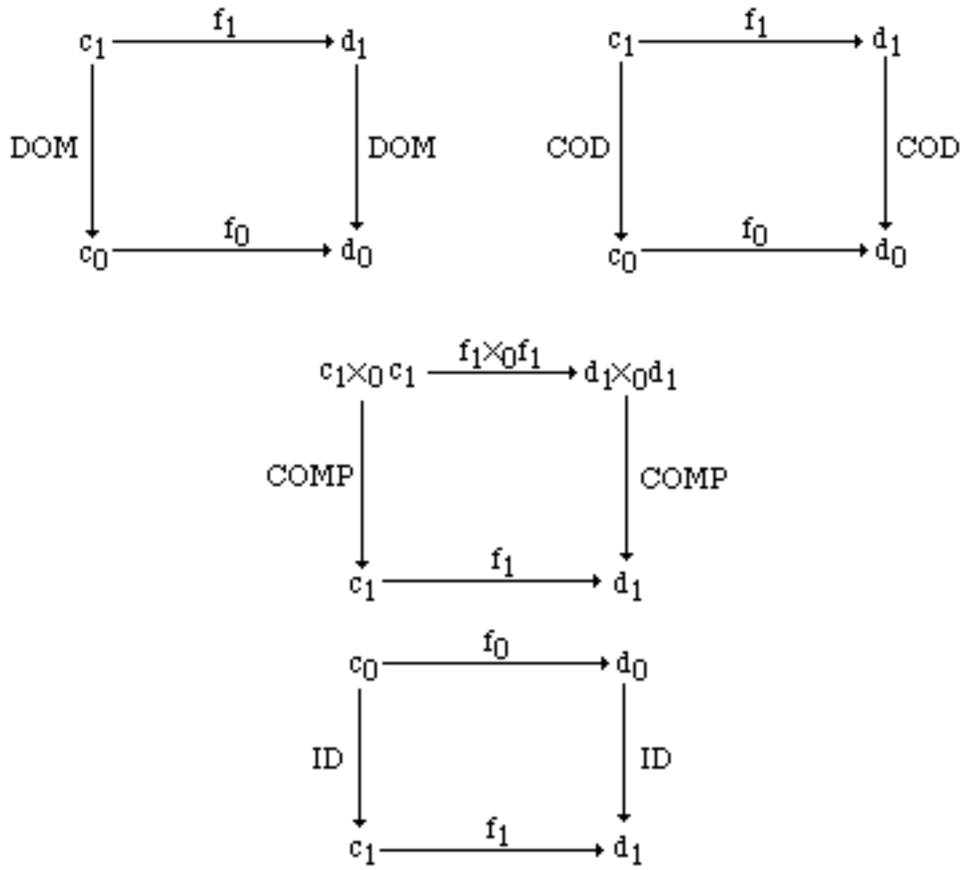
$$p_1{}^*(f) = f \times id: U \times C \to C \times C$$

$$p_2{}^*(f) = id \times f: C \times U \to C \times C$$

are respectively the pullbaks of f along the first and second projections. Composition is obtained from the internal composition map $m: p_2{}^*(f)^{p_1{}^*(f)} \times p_3{}^*(f)^{p_2{}^*(f)} \to p_3{}^*(f)^{p_1{}^*(f)}$ in the slice category $E/C \times C \times C$. Similarly, the identity morphism $ID: C \to Full(f)_1$ is obtained from the "inclusion of identities" $\Lambda(id_f): id_C \to f^f$ in the slice category E/C.

Our exposition of Internal Category Theory proceeds with the definition of "internal functor." Again, the intuition of a standard functor helps in the understanding of the following definition; a functor F between two small categories C and D is a pair of functions in **Set**, $F = (F_0, F_1)$, where $F_0: Ob_C \to Ob_D$, $F_1: Mor_C \to Mor_D$; moreover F_1 distributes with respect to composition and preserves identity.

7.2.3 Definition *Let $c,d \in Cat(E)$. F is an* ***internal functor*** *from c to d ($F: c \to d$) iff $F = (f_0, f_1)$ with $f_0 \in E[c_0,d_0]$, $f_1 \in E[c_1, d_1]$, and F satisfies*



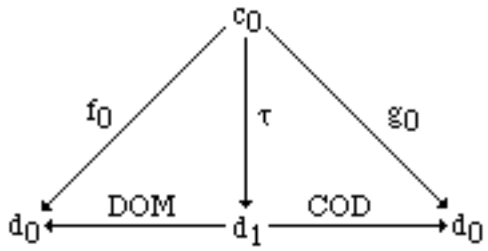
7.2.4 Definition *The category* ***Cat(E)*** *has as objects the internal categories of E and as morphisms the internal functors. Composition of functors is defined in the obvious way; that is, given* $F = (f_0,f_1)$ *and* $G = (g_0,g_1)$, $F \circ G = (f_0 \circ g_0, f_1 \circ g_1)$.

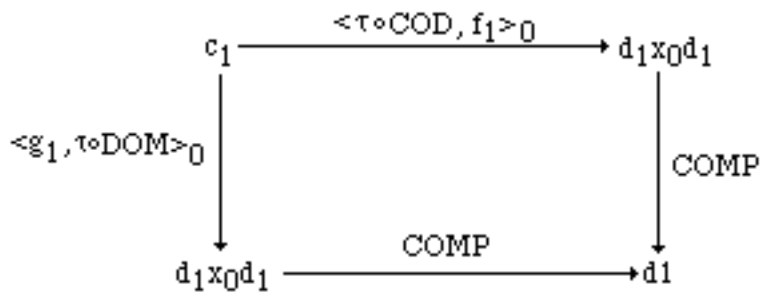
For example, Cat(**Set**) is the category **Cat** of all small categories, i.e., of all those categories whose class of morphisms is a set.

It is easy to carry out the usual constructions on categories inside **Cat(E)**. For example, given c = (c_0, c_1, DOM, COD, COMP, ID), we can define the **dual** category c^{op} = (c_0, c_1, COD, DOM, COMP∘α, ID), where $\alpha = \langle\Pi_2,\Pi_1\rangle_0 : c_1\times_0 c_1 \leftrightarrow c_1\times_0' c_1$. $\_^{op}$: **Cat(E)** → **Cat(E)** is a functor.

The **product** of two internal categories c and d is the category c×d = ($c_0\times d_0$, $c_1\times d_1$, $DOM_c\times DOM_d$, $COD_c\times COD_d$, $(COMP_c\times COMP_d) \circ \beta$, $ID_c\times ID_d$) where β is the isomorphism $(c_1\times_0 c_1)\times(d_1\times_0 d_1) \leftrightarrow (c_1\times d_1)\times_0(c_1\times d_1)$. Clearly, _×_ : **Cat(E)**×**Cat(E)** → **Cat(E)** is a functor.

7.2.5 Definition *Let* $F = (f_0,f_1)$ *and* $G = (g_0,g_1)$ *be two internal functors from* c *to* d. τ *is an* ***internal natural transformation*** *from* F *to* G (τ: F→G) *iff* $\tau\in E[c_0, d_1]$ *and satisfies*





7.2.6 Definition *Given two internal categories c and d,* ***Nat(c,d)*** *is the category that has internal functors from c to d as objects, and internal natural transformations as arrows. Given* σ*:* $F \to G$ *and* $\tau: G \to H$*,* $\tau \circ \sigma = COMP_d \circ <\tau,\sigma>_0 : F \to H$

7.2.7 Example In this example we define **PER** as an internal category of ω-Set. **PER** is the category of partial equivalence relations constructed over Kleene's applicative structure (ω,·). Remember that the partial application $\cdot : \omega\times\omega \to \omega$ is defined by $m\cdot n = \varphi_m(n)$, where $\varphi: \omega \to PR$ is an acceptable gödel numbering of the partial recursive functions. We will use the following notation:

n A m iff n is related to m by A ,

$\{n\}_A$ = {m | m A n } the equivalence class of n with respect to A,

Q(A) = $\{\{n\}_A \mid n \in dom(A)\}$ where dom(A) = {n/ nAn} .

The morphisms of the category are defined by

f∈PER[A,B] iff f : Q(A)→Q(B) and ∃n ∀p (pAp ⇒ f({p}A) = {n.p}B) .

Thus the morphisms in **PER** are "computable" in the sense that they are fully described by partial recursive functions, which are total on the domain of the source relation.

Note that **PER** is a small category, as the partial equivalence relations (p.e.r.'s) form a set as well as their morphisms; thus **Set** contains **PER** as an internal category. Though, since a crucial property of **PER** is that its morphisms are "computable," we are interested in introducing a similar notion in the category of sets by a **realizability** relation "⊩" with respect to numbers.

The category **ω−Set** is defined as follows:

objects: (A,⊩)∈**ω−Set** iff

A is a set and ⊩ ⊆ ω×A, such that ∀a∈A ∃n ⊩ a .

morphisms : f∈**ω−Set**[A,B] iff

$f : A \to B$ and $\exists n\ \forall a \in A\ \forall p \vdash_A a \quad n \cdot p \vdash_B f(a)$

(*notation* : $n \vdash_{A \to B} f$ and we say that n **realizes** f).

Similarly as for **PER**, each morphism in **ω−Set** is "computed" by a partial recursive function, which is total on $\{ p \mid p \vdash_A a \}$ for each $a \in A$.

It is not difficult to prove that **ω−Set** is a CCC with all finite limits. The terminal object is simply $(\mathbf{1}, \vdash_1)$, where **1** is the singleton set and $\vdash_1 = \omega \times \mathbf{1}$. If [,] is a coding of pairs of numbers, then $(A \times B, \vdash_{A \times B})$ is given by $[n,m] \vdash_{A \times B} (a,b)$ iff $n \vdash_A a$ and $m \vdash_B b$. As for exponents, let $[A \to B] = (\{f: A \to B \mid f \in \textbf{ω-Set}[A,B]\}, \vdash_{A \to B})$, where $\vdash_{A \to B}$ is given as above.

There is a simple way to embed **Set** into **ω-Set**. Let $\Sigma : \textbf{Set} \to \textbf{ω-Set}$ be given by

$$\Sigma(S) = (S, \vdash_S) \quad \text{with} \quad \vdash_S = \omega \times S \text{, the "full" relation.}$$

Σ is defined as the identity on morphisms, since by the definition of $\vdash_S$, all functions are realized by all numbers for total recursive functions. Σ is a full and faithful functor, which preserves all finite limits and exponents.

This embedding suggests how to turn **PER** into an internal category of **ω-Set** (recall that the exponent of A and B in **PER** is given by $m\ (A \to B)\ n \iff \forall p,q\ (p\ A\ q \Rightarrow m \cdot p\ B\ n \cdot q)$). Indeed, $\mathbf{M} = (\mathbf{M_0}, \mathbf{M_1}, dom^M, cod^M, id^M, comp^M)$ is defined by

1. $\mathbf{M_0} = (\mathbf{PER}, \vdash_M)$ where $\vdash_M = \omega \times \mathbf{PER}$;
2. $\mathbf{M_1} = (\{<\{n\}_{A \to B}, A, B)> \mid A, B \in \mathbf{M},\ n\ (A \to B)\ n\}, \vdash_1)$
   where $m \vdash_1 <\{n\}_{A \to B}, A, B>$ iff $m\ (A \to B)\ n$;
3. $dom^M(<\{n\}_{A \to B}, A, B)>) = A$;
4. $cod^M(<\{n\}_{A \to B}, A, B)>) = B$;
5. $id^M(A) = <\{i\}_{A \to A}, A, A)>$ where $i = \lambda x.x$ is a number for the identity function;
6. $comp^M(<\{n\}_{A \to B}, A, B)>, <\{m\}_{B \to C}, B, C>) = <\{b \cdot m \cdot n\}_{A \to C}, A, C)>$
   where $b = \lambda xyz.x(yz)$.

We have to check that **M** is an internal category of **ω-Set**. It will be easy, in view of the set-theoretic nature of its morphisms. Essentially, one has to prove that the required morphisms are functions that happen to be realized.

Note first that $\textbf{ω-Set}[\underline{A}, \Sigma(S)] = \textbf{Set}[A,S]$ for any $\underline{A} = (A, \vdash_A)$ in **ω-Set** and any set S, since $\vdash_S$ is the full relation and, hence, any function is realized by any index. Thus, the set-theoretic functions dom^M, cod^M are also morphisms in **ω-Set**.

$\mathbf{M_1}$ is a set of triples: equivalence class, domain, and codomain. The realizability relation in $\mathbf{M_1}$ is nontrivial and, hence, one needs to give explicitly the realizers of id^M and $comp^M$. Indeed, id^M is realized by $\underline{\lambda}x.i$, the constant function equal to an index i for the identity function. As for $comp^M$, it is defined as usual only on a subset of $\mathbf{M_1} \times \mathbf{M_1}$, namely, where the target of the first morphism coincides with the source of the second. In the general setting, this is expressed by the use of a pullback as a source object for COMP. In this specific case, that pullback becomes simply the set of pairs such as $(<\{n\}_{A\to B},A,B)>,<\{m\}_{B\to C},B,C>)$. Then the realizer for $comp^M$ is b', for b'[n,m] = bnm, where b is an index for the composition of functions, an operation that may be uniformly and effectively given over $(\omega,\cdot)$.

7.3 Internal Presheaves

We have already remarked that every small category may be regarded as an internal category in **Set.** However, in **Set** we are accustomed to considering not only functors from one small category to another, but also, for example, functors from a small category to a large one and in particular to **Set** itself. A significant example is hom-functor from a small category to **Set**. Surprisingly, it is possible to cope at the internal level also with this problem, by means of the notion of **internal presheaf**.

If F is a functor from $\mathbf{C}^{op}$ to **Set**, then the component F_{Ob} of F is a collection {F(c)} of sets indexed on objects of **C**. Such a collection can be regarded as a function ρ_0: $X\to Ob_{\mathbf{C}}$, where $X = \{(c,m)/\ m\in F(c)\}$ and $\rho_0(c,m) = c$. Then $F_{Ob}(c) \cong \rho_0^{-1}(c)$. Now, given an arrow f: d→c, and an object $(c,m)\in\rho_0^{-1}(c)$, define a function ρ_1 by $\rho_1((c,m),f) = (d,F(f)(m))$. The function ρ_1 describes the behavior of F on morphisms. Note that $\rho_1((c,m),f)$ is defined if and only if cod(f) = $\rho_0(c,m) = c$; thus, the domain of ρ_1 is the pullback Z (in **Set**) of cod: $Mor_{\mathbf{C}}\to Ob_{\mathbf{C}}$ and ρ_0: $X\to Ob_{\mathbf{C}}$. Let Π_2: $Z\to Mor_{\mathbf{C}}$ and Π_1: $Z\to X$, be the associated projections. Note that

1. $\rho_0(\ \rho_1((c,m), f)\ ) = \rho_0(\ (d,F(f)(m))\ ) = d = dom(f) = dom(\ \Pi_2(f,(c,m))\ )$;
2. $\rho_1((c,m),f\circ f') = (d,F(f\circ f')(m)) = (d,F(f')(F(f)(m))) = \rho_1((d',F(f)(m)),f') =$
   $= \rho_1(\rho_1(f,(c,m)),f')$;
3. $\rho_1((c,m), id_c) = (c,F(id_c)(m)) = (c, m)$.

That is, more concisely:

i. $\rho_0 \circ \rho_1 = dom \circ \Pi_2 : Z\to Ob_{\mathbf{C}}$;

ii. $\rho_1 \circ (id_X \times_0 comp) = \rho_1 \circ (\ \rho_1 \times_0 id_{Mor_{\mathbf{C}}}) : X\times_0 Mor_{\mathbf{C}}\times_0 Mor_{\mathbf{C}} \to Mor_{\mathbf{C}}$;

iii. $\rho_1 \circ <id_X, ID \circ \rho_0>_0 = id_X$,

where $\times_0$ denotes pullback product and ID: $Ob_{\mathbf{C}}\to Mor_{\mathbf{C}}$ is the function that takes an object c to id_c. Conversely, given a small category **C**, and a triple $(X,\ \rho_0: X\to Ob_{\mathbf{C}},\ \rho_1: X\times_0 Ob_{\mathbf{C}}\to Mor_{\mathbf{C}}\ )$ that satisfies equations i-iii above, it is possible to define a presheaf F: $\mathbf{C}^{op}\to$**Set** by letting

$\forall c\in Ob_{\mathbf{C}}$ $\qquad$ $F(c) = \rho_0^{-1}(c)$

$\forall f\in \mathbf{C}[c',c],\ \forall(c,m)\in F(c)$ $\qquad$ $F(f)(c,m) = \rho_1((c,m),f)$.

Equation i states that $\rho_1((c,m),f)$ is in F(c'), indeed $c' = dom(f) = dom(\Pi_2((c,m),f)) = \rho_0(\rho_1((c,m),f))$, and thus, by definition of F, $F(f)(c,m)=\rho_1((c,m),f)\in F(c')$.

Equations ii and iii express the fact that F is a contravariant functor. Indeed,

$$F(f\circ g)(c,m) = \rho_1((c,m),comp(f,g)) \quad \text{by def. of F}$$
$$= \rho_1(\ \rho_1((c,m),f),\ g) \quad \text{by (ii)}$$

$= F(g)(\ \rho_1((c,m),f))$ by def. of F

$= F(g)(F(f)((c,m)))$ by def. of F

and

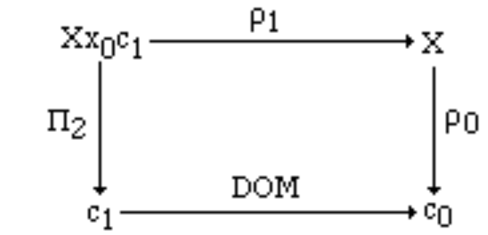
$F(id_c)(c,m) = \rho_1((c,m), id_c)$ by def. of F

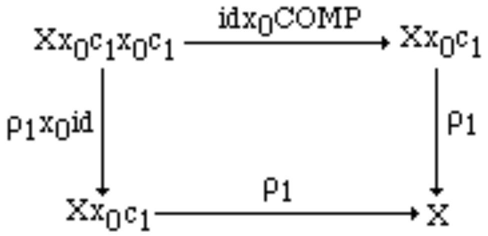
$= (c,m)$ by (iii)

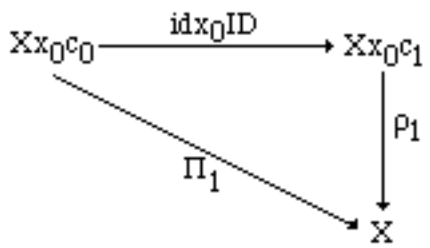
7.3.1 Definition *X is an* ***internal presheaf*** *on* $c \in Cat(E)$ *iff* $X = (X, \rho_0, \rho_1)$ *with,*

$\rho_0: X \to c_0$

$\rho_1: X \times_0 c_1 \to X$ *where* $X \times_0 c_1$ *is the pullback of* $\rho_0: X \to c_0$ *and* $COD: c_1 \to c_0$,
and X satisfies the following:







Example Let $c \in Cat(E)$, and e an object of E. The **constant-e** diagram is the internal presheaves ($e \times c_0$, snd: $e \times c_0 \to c_0$, $id_e \times DOM$: $e \times c_1 \to e \times c_0$). Note that $e \times c_1$ is the pullback of snd: $e \times c_0 \to c_0$ and COD: $c_1 \to c_0$. Moreover, the previous morphism satisfies the requested conditions of definition 7.3.1, since

i. $snd \circ id_e \times DOM = DOM \circ snd : e \times c_1 \to c_0$;

ii. $id_e \times DOM \circ (id_e \times COMP) =$

$= id_e \times (DOM \circ COMP)$

$= id_e \times (DOM \circ \Pi_2)$

$= id_e \times DOM \circ (id_e \times \Pi_2)$
$= id_e \times DOM \circ (id_e \times DOM \times_0 id) : e \times c_1 \times_0 c_1 \to e \times c_0$;

iii. $id_e \times DOM \circ (id_e \times ID) = id_{e \times c_0} : e \times c_0 \to e \times c_0$.

The intuition behind the previous definition is that of a collection, indexed by c, of objects e. Indeed, consider the case of an internal category **C** in **Set** (i.e., a small category) and let E be a set. By applying the above "externalization," we obtain

$\forall c \in Ob_{\mathbf{C}}$ $\qquad F(c) = \rho_0^{-1}(c) = E \times \{c\}$

$\forall f \in \mathbf{C}[c',c],\ \forall (e,c) \in F(c)$ $\qquad F(f)(e,c) = \rho_1((e,c),f) = (e,DOM(f)) = (e,c')$.

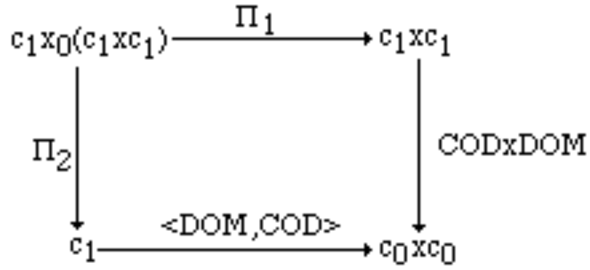
Another major example of a presheaf is given by the hom-functor.

7.3.2 Definition *Let $c \in Cat(E)$. The* ***internal hom-functor*** *hom_c is the presheaf (c_1, ρ_0, ρ_1) on $c \times c^{op}$, where*

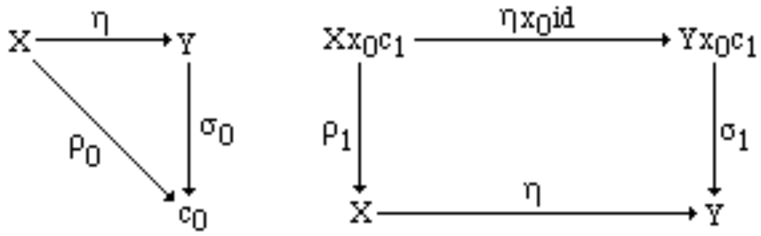
$\rho_0 = \langle DOM,COD \rangle : c_1 \to c_0 \times c_0$

$\rho_1 = COMP \circ \langle p_2 \circ \Pi_1, COMP \circ (id \times_0 p_1) \rangle_0 : c_1 \times_0 (c_1 \times c_1) \to c_1$

(Informally, $\rho_1 = \lambda fgh.\ h \circ f \circ g$), and

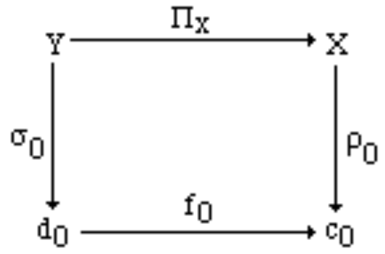


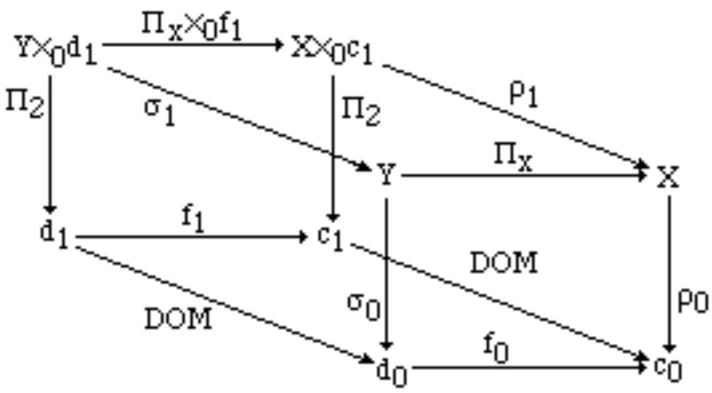
7.3.3 Definition *Let $X = (X, \rho_0 ,\rho_1)$, $Y = (Y, \sigma_0 ,\sigma_1)$ be two presheaves on $c \in Cat(E)$. η is a* ***morphism of presheaves*** *from X to Y ($\eta : X \to Y$) iff $\eta \in E[X,Y]$ and the following diagrams commute:*



The following definition allows to compose an internal presheaf on c with an internal functor F: $d \to c$, yielding a new presheaf on d.

7.3.4 Definition *Let $X = (X, \rho_0, \rho_1)$ be an internal presheaf on $c \in Cat(E)$, and $F: d \to c$ be an internal functor. The* ***pullback of X along F*** *is the presheaf $F^*(X) = (Y, \sigma_0, \sigma_1)$ on d defined by the following commutative diagrams, where the squares are pullbacks:*





Suppose that the internal presheaf X "internalizes" the functor G: $\mathbf{C}^{op}\to\mathbf{Set}$ (and F: d→c is an "internalization" for F: $\mathbf{D}\to\mathbf{C}$). Then, $G(F(a)) = \{x\in X \mid \rho_0(x)=f_0(a)\} = \{y\in Y \mid \sigma_0(y)=a\}$ by definition of the pullback for Y, and, if h: a→b, one has $G(F(h)) = \lambda x\in F(b).\rho_1(x,f_1(h)) = \lambda x\in F(b).\sigma_1(x,h)$ by definition of σ_1.

All the definitions given so far were directed towards the following crucial notion, which will enable us to define the concept of internal Cartesian closed category.

7.3.5 Definition *< F, G, φ > : c→d is an* ***internal adjunction*** *from c to d iff F is an internal functor from c to d, G is an internal functor from d to c and*

$$\phi : (F\times Id_d{}^{op})^*(hom_d) \to (Id_c\times G^{op})^*(hom_c)$$

is an isomorphism between presheaves on $c\times d^{op}$.

The definition of adjunction in 7.3.5 is now easily generalized to the case with parameters.

7.3.6 Definition *< F, G, φ > : c→d is an* ***internal adjunction*** *from c to d* ***with parameters*** *in a iff F is an internal functor from c×a in d, G is an internal functor from* $a^{op}\times d$ *in c and*

$$\phi : (F\times Id_d{}^{op})^*(hom_d) \to (Id_c\times G^{op})^*(hom_c)$$

is an isomorphism between presheaves on $c\times a\times d^{op}$.

We can also give an "equational" characterization of internal adjunctions, in the spirit of theorem 5.3.5.

7.3.7 Theorem *Every internal adjunction* $< F, G, \phi > : c \to d$ *is fully determined by the following data in (i) or (ii):*

i. - *the functor* $G: d \to c$
- *an arrow* $f_0: c_0 \to d_0$
- *an arrow* $Unit: c_0 \to c_1$ *such that* $DOM \circ Unit = id$, $COD \circ Unit = g_0 \circ f_0$
- *an arrow* $\phi^{-1}: Y \to X$, *where X and Y are respectively the pullbacks of*

$<DOM,COD> : d_1 \to d_0 \times d_0$, $f_0 \times id: c_0 \times d_0 \to d_0 \times d_0$

$<DOM,COD> : c_1 \to c_0 \times c_0$, $id \times g_0: c_0 \times d_0 \to c_0 \times c_0$

and, moreover, the previous functions satisfy the following equations:

a. $< \rho_0, COMP \circ < g_1 \circ \Pi_X, Unit \circ p_1 \circ \rho_0 >_0 >_0 \circ \phi^{-1} = id_Y$

b. $\phi^{-1} \circ < \rho_0, COMP \circ < g_1 \circ \Pi_X, Unit \circ p_1 \circ \rho_0 >_0 >_0 = id_X$

ii. - *the functor* $F: c \to d$,
- *an arrow* $g_0: d_0 \to c_0$,
- *an arrow* $Counit: d_0 \to d_1$ *such that* $DOM \circ Counit = f_0 \circ g_0$, $COD \circ Counit = id$
- *an arrow* $\phi: X \to Y$, *where X and Y are respectively the pullbacks of*

$<DOM,COD> : d_1 \to d_0 \times d_0$, $f_0 \times id : c_0 \times d_0 \to d_0 \times d_0$

$<DOM,COD> : c_1 \to c_0 \times c_0$, $id \times g_0 : c_0 \times d_0 \to c_0 \times c_0$

and moreover the previous functions satisfy the following equations:

a. $< \rho_0', COMP \circ < Counit \circ p_2 \circ \rho_0', f_1 \circ \Pi_Y >_0 >_0 \circ \phi = id_X$

b. $\phi \circ < \rho_0', COMP \circ < Counit \circ p_2 \circ \rho_0', f_1 \circ \Pi_Y >_0 >_0 = id_Y$

Proof See the appendix to this chapter.

We are finally ready to define internal Cartesian closed categories.

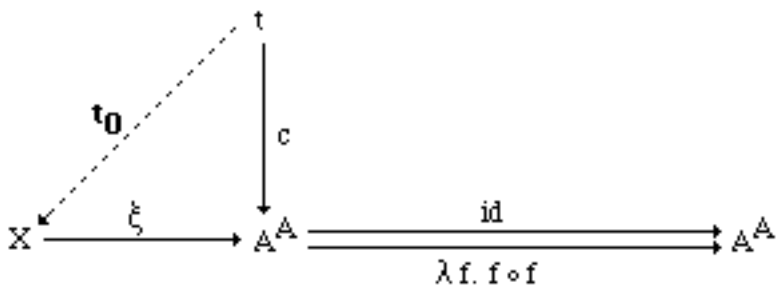
7.3.8 Definition *An* ***internal Cartesian closed category*** *is a category* $c \in Cat(E)$ *with three adjunctions, the third one with parameter in c:*

1. $< O, T, \mathbb{0} > : c \to 1$, where 1 is the internal terminal category.
2. $< \Delta, x, \langle\!\langle,\rangle\!\rangle > : c \to c \times c$, where Δ is the internal diagonal functor.
3. $< x, [,] , \mathbb{A} > : c \to c$, where this adjunction has parameters in c.

7.3.9 Examples 1. In example 2 in 7.2.2, we defined the internal category $\mathbf{Ret_A} \in Cat(E)$ of retractions on a generic object A of E, where E is a CCC with all finite limits. We now prove that if A is a reflexive object, that is, if $A^A < A$, then $\mathbf{Ret_A}$ is Cartesian closed.

Let $A^A < A$ via (in,out). By theorem 2.3.6 we know that $t < A$ and $A\times A < A$. Call these retractions (in',out') and (in",out"), respectively.

Let us begin with the internal terminal object in $\mathbf{Ret_A}$. The idea is that every constant function is a terminal object in a category of retractions. Since $t < A$ via (in',out'), in': $t \to A$ is a point of A and, thus, we can take in'∘ out': $A \to A$ as the constant function we are looking for; moreover, $c = \Lambda(\mathrm{in}'\circ \mathrm{out}'\circ p_2) : t \to A^A$ is the point in A^A that represent it. Then the internal terminal object $\mathbf{t_0}$: $t \to X$ is defined by the following:

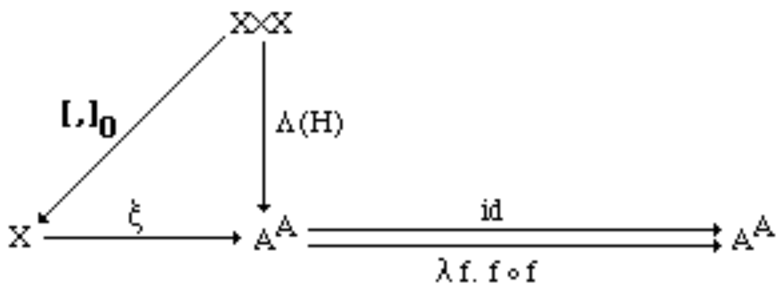


We leave it to the reader to check the soundness of the previous definition, as well as the definition of internal products, and we move on to exponents.

The first notion we must define is the arrow $\mathbf{[,]_0}$: $X\times X \to X$. The idea is that, given two retractions f, g, their exponent is the retraction $\mathbf{[f,g]_0}=\lambda a.\mathrm{in}(\xi(g)\circ \mathrm{out}(a)\circ\xi(f))$. Let

$$H = \lambda(f,g)\lambda a.\ \mathrm{in}(\ \xi(g)\circ \mathrm{out}(a)\circ\xi(f)\ ) :\ (X\times X)\times A \to A\ .$$

Then $\mathbf{[,]_0}$: $X\times X \to X$ is formally defined by the following diagram:



The function **EVAL**: $X\times X \to Y$ is the internal Counit of the adjunction; it takes two retractions f and g, and gives a morphism $\mathbf{EVAL}_{f,g}$ from the retraction $\mathbf{[f,g]_0 x_0} f$ to the retraction g (where $\mathbf{x_0}$ is the internal product on objects). More specifically, if

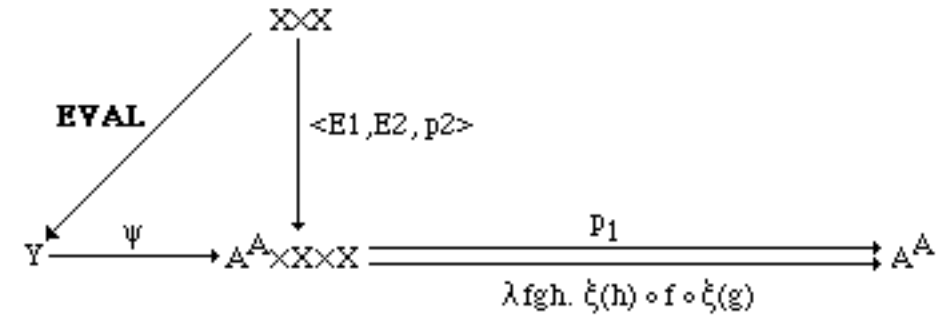
$E = \lambda(f,g)\lambda a{:}\mathbf{[f,g]_0x_0}f.\ \mathrm{out}(\mathbf{FST}(a))(\mathbf{SND}(a)) : X\times X\times A \to A$

(where $\lambda a{:}h.M$ is shorthand for $\lambda a.[h(a)/a]M$, and **FST**, **SND** are the internal projections)

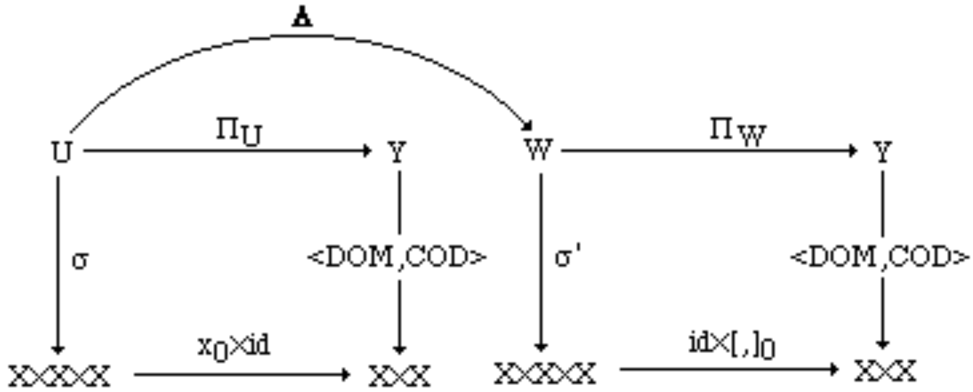
$E_1 = \Lambda(E): X\times X \to A^A$

$E_2 = \mathbf{x_0}\circ\langle \mathbf{[,]_0}, p_1\rangle: X\times X \to X.$

Then **EVAL:** $X\times X \to Y$ is defined by the following commutative diagram:



We must now define **Λ**: U → W, where U and W are the pullbacks in the following diagram:



Informally **Λ** works on tuples of the kind (f,g,h, (r, f$\mathbf{x_0}$g, h)) where f,g,h are retractions and r is a morphism from f$\mathbf{x_0}$g to h, that is r: A→A such that r = h ∘ r ∘ f$\mathbf{x_0}$g.

Now, let Curry(r) = λy.in(λz.(r ∘ in")(z,y)): A→A. Then Curry(r) is a morphism from g to $\mathbf{[f,h]_0}$: indeed, by omitting for simplicity the function ξ: X→A^A, we have

$\mathbf{[f,h]_0}$ ∘ λy.in(λz.(r ∘ in")(z,y)) ∘ g =

= λa.in(h ∘ out(a) ∘ f) ∘ λy.in(λz.(r ∘ in")(z,g(y)))

= λy.in(h ∘ λz.(r ∘ in")(z,g(y)) ∘ f)

= λy.in(λz.(h ∘ r ∘ in")(f(z),g(y)))

= λy.in(λz.(h ∘ r ∘ f$\mathbf{x_0}$g ∘ in")(z,y))

= λy.in(λz.(r ∘ in")(z,y))

Let Curry = λr.λy.in(λz.(r ∘ in")(z,y)): A^A →A^A.

Then F = < Curry ∘ p_1 ∘ ψ ∘ Π_U, id×$\mathbf{[,]_0}$ ∘ σ > : U→A^A×X×X .

But we have already verified that

p_1 ∘ F = (λfgh. ξ(h) ∘ f ∘ ξ(g)) ∘ F

and, thus, there exists a unique morphism **F**: U→Y such that F = ψ ∘ **F**.

Finally **Λ**= <s, **F**>$_0$: U → W.

2. This example continues example 7.2.7, where we defined **PER** as an internal category of the category **ω-Set**. We still need to check that the internal category **PER** of **ω-Set** is an internal CCC. In general, observe that in order to "internalize" a categorical construction, as we did for the category

of retractions, say, one has to turn implicit set-theoretic functional dependencies into morphisms of the intended global category **E**. For example, consider the map **Λ** that gives the internal natural isomorphism for Cartesian closure. Externally, **Λ** is implicitly indexed by objects a, b, for instance, and the map a,b ⊢ $\Lambda_{a,b}$ is simply a function in **Set**. The internal version, requires only that the map **Λ**, depending also on a and b, is a morphism in **E**.

The result, that **M** is an internal CCC of **ω-Set**, then follows by the uniformity and effectiveness of the argument for the Cartesian closure of **PER**. Namely, one only has to observe that $eval_{A,B}$ is realized by any index e of the partial recursive universal function (and hence we could set $e_{A,B} = e$ in the example). Thus, not only $eval_{A,B}$ is realized, but the construction is internal to **ω-Set** as it depends on A,B by a constant function (or e is independent of A, B). Thuis is also the case for $\Lambda_{A,B}$, since it is uniformly realized by any index of the function s of the s-m-n iteration theorem, independently of A, B.

7.4 Externalization

In this section, we define the process of *externalization* of an internal category via hom-functors that correspond, essentially, to the Yoneda embedding. Since for any object e of E the hom functor [e,_]: E→**Set** preserves pullbacks, it transforms an internal category $c = (c_0, c_1$, DOM, COD, COMP, ID)∈Cat(E) into a small category [e,c] = ([e,c_0], [e,c_1], [e,DOM], [e,COD], [e,COMP], [e,ID]).

More generally, if c∈Cat(E), then [_,c]∈Cat(E^{op}→**Set**), and, for the uniform behavior with respect to the indexes in E, [_,c] can be also regarded as an E-indexed category, that is, a functor E^{op}→**Cat**. In the next section we show that, conversely, every E-indexed category can be regarded as an internal category in E^{op}→**Set**.

7.4.1 Definition *Let c = (c_0, c_1, DOM, COD, COMP, ID) ∈ Cat(E), then [e,c] = ([e,c_0], [e,c_1], [e,DOM], [e,COD], [e,COMP], [e,ID]).*

The objects of [e,c] are the arrows σ∈ E[e, c_0]. Given two objects σ, τ, a morphism f: σ→τ in [e,c] is an arrow f∈E[e, c_1] such that DOM ∘ f = σ, COD ∘ f = τ . The identity of σ is id_σ = ID ∘ σ. Let c_2 be the object of composable maps of c, that is the pullback $c_1 \times_0 c_1$ of COD and DOM. Since the hom-functor [e,_]: E→**Set** preserves pullbacks, [e,c_2] is the pullback of [e,COD] and [e,DOM], and [e,COMP]: [e,c_2]→[e,c_1] has the expected type. Given two arrows f: σ→τ, g: τ→γ in [e,c], their composition by [e,COMP] is g ◎ f = COMP ∘ $<g,f>_0$. In case the ambient category E has small hom-sets, the category [e,c] is obviously small.

Note that, if c,d∈Cat(E), then [e,c×d] ≅ [e,c]×[e,d] and [e,c^{op}] ≅ $[e,c]^{op}$.

In the previous definition, e can be regarded as a parameter, yielding a functor [_,c] : E^{op}→**Cat**, that is, an E-indexed category.

7.4.2 Definition *Let c∈Cat(E). The functor [_,c] : E^{op}→Cat is defined in the following way:*

on objects e∈E *[_,c] = [e,c]*

on arrows σ: e'→e *[_,c](σ) = [σ,c] is the functor from [e,c] in [e',c] that is defined as $[\sigma,c_0]$ on objects and as $[\sigma,c_1]$ on arrows.*

More explicitly, the functor [σ,c] takes every τ∈[e,c] (i.e., τ: e→c_0) to τ ∘ σ, and every g: τ→τ' to g ∘ σ.

We have to prove as follows that the previous definition makes sense:

1. ∀σ: e'→e, [σ,c]: [e,c] → [e',c] is a functor, for
   1.1. ∀τ: e→c_0 $[\sigma,c](id_\tau) = [\sigma,c](ID \circ \tau) = ID \circ \tau \circ \sigma = id_{\tau \circ \sigma}$
   1.2. ∀f: δ→γ , ∀g: ρ→δ in [e,c]
   [σ,c](f ⊚ g) = COMP ∘ < f, g > ∘ σ = COMP ∘ < f ∘ σ, g ∘ σ > = [σ,c](f) ⊚ [σ,c](g)
2. [_,c] : E^{op} → Cat is a functor, for
   2.1. ∀e [_,c] (id_e) = I : [e,c] → [e,c] (immediate by definition of [_,c])
   2.2. ∀σ: e→e' , ∀τ: e'→e", [_,c](τ ∘ σ) = [_,c](σ) ∘ [_,c](τ) : [e,c] → [e",c]; indeed,
      2.2.1. on objects γ∈[e,c]: [_,c](τ ∘ σ)(γ) = γ ∘ τ ∘ σ = [_,c](σ)([_,c](τ)(γ))
      2.2.1. on arrows g τ→τ' in [_,c]: [_,c](τ ∘ σ)(g) = g ∘ τ ∘ σ = [_,c](σ)([_,c](τ)(g))

Note that if c = (c_0, c_1, DOM, COD, COMP, ID) is an internal category in E, then ([_,c_0], [_,c_1], [_,DOM], [_,COD], [_,COMP], [_,ID]) is an internal category in E^{op}→**Set**.

Definitions 7.4.3 and 7.4.4 show how to externalize, respectively, an internal functor, an internal natural transformation, and an internal presheaf. Again these definitions, as well as others in the sequel, are parametric with respect to the object e of E .

7.4.3 Definition *Let c,d∈Cat(E), F = (f_0,f_1): c→d be an internal functor, and let e be an object of E. The functor [e,F]: [e,c]→ [e,d] is defined as [e,f_0] on objects, and as [e,f_1] on arrows.*

That is, the functor [e,F]: [e,c] → [e,d] takes every object σ in [e,c] to f_0 ∘ σ in [e,d], and every arrow g: σ→τ in [e,c] to f_1∘ g: (f_0 ∘ σ)→(f_0 ∘ τ) in [e,d].

7.4.4 Definition *Let $c,d \in Cat(E)$ and let $F = (f_0,f_1): c \to d$ be an internal functor. The E-indexed functor $[\_,F]: [\_,c] \to [\_,d]$ is the natural transformation defined by $[\_,F](e) = [e,F]$, for every object e of E.*

We must prove the naturality in e of the previous definition; that is, for any $\sigma: e' \to e$,

$$[e',F] \circ [\sigma,c] = [\sigma,d] \circ [e,F] .$$

We have, for any object τ of $[e,c]$ (i.e, $\tau: e \to c_0$),

$$\begin{aligned} [e,F][\sigma,c](\tau) &= [e',F](\tau \circ \sigma) && \text{by def. of } [\sigma,c] \\ &= f_0 \circ \tau \circ \sigma && \text{by def. of } [e',F] \\ &= [\sigma,d] \circ f_0 \circ \tau && \text{by def. of } [\sigma,d] \\ &= [\sigma,d] \circ [e,F] && \text{by def. of } [e,F]. \end{aligned}$$

7.4.5 Definition *Let $\tau: F \to G$ be an internal natural transformation, where $F,G: c \to d$. The natural transformation $[e,\tau]: [e,F] \to [e,G]$ is defined as the homonimous function $[e,\tau]: [e,c_0] \to [e,d_1]$; that is, it takes every object σ of $[e,c]$ to $[e,\tau](\sigma) = \tau \circ \sigma : (f_0 \circ \sigma) \to (g_0 \circ \sigma)$ (where the last "typing" is in $[e,c]$).*

Exercise Prove that the previous definition makes sense, that is:

1. $[e,\tau](\sigma) : [e,F](\sigma) \to [e,G](\sigma)$
2. for every $h: \sigma \to \gamma$ in $[e,c]$, $[e,G](h) \circ [e,\tau](\sigma) = [e,\tau](\gamma) \circ [e,F](h)$.

7.4.6 Definition *Let $\tau: F \to G$ be an internal natural transformation, where $F,G: c \to d$. The E-indexed natural transformation $[\_,\tau]: [\_,F] \to [\_,G]$ is defined by the following: for any object e of E, $[e,\tau]: [e,F] \to [e,G]$.*

Now we will show how to externalize the notion of *morphism of presheaves*.

7.4.7 Definition *Let $X=(X, \rho_0, \rho_1)$ be an internal presheaf on $c \in Cat(E)$. The functor $[e,X]: [e,c]^{op} \to$ **Set** is defined by:*

$\forall \sigma \in [e,c]$, $\quad [e,X](\sigma) = \{ f \in E[e,X] \ / \ \rho_0 \circ f = \sigma \}$

$\forall g: \tau \to \sigma$ in $[e,c]$, $\quad [e,X](g) : [e,X](\sigma) \to [e,X](\tau)$ is given by:

$\forall f \in [e,X](\sigma) \quad [e,X](g)(f) = \rho_1 \circ \langle f, g\rangle_0 \in [e,X](\tau)$

(note that $\rho_0 \circ [e,X](g)(f) = \rho_0 \circ \rho_1 \circ \langle f,g\rangle_0 = DOM \circ \Pi_2 \circ \langle f,g\rangle_0 = DOM \circ g = \tau$)

We have chosen the name $[e,X]$ as an analogy for the previous constructions, but in this case it no longer has a direct relation with the Yoneda embedding. The same holds below for the externalization $[e,\eta]$ of a morphism of presheaves η.

Next we check that by externalizing an internal hom_c on $c\times c^{op}$ we just obtain the hom-functor from $[e,c]^{op}\times[e,c]$ to **Set**.

7.4.8 Proposition *Let $c\in Cat(E)$ and let $hom_c = (c_1, \rho_0, \rho_1)$ be the internal hom-functor on $c\times c^{op}$. Then, for every $e\in Ob_E$, $[e,hom_c] = hom_{[e,c]}: [e,c]^{op}\times[e,c]\to$ **Set** (to within the implicit isomorphism $[e,c]^{op}\times[e,c] \cong [e,c^{op}\times c]$).*

Proof

- on objects: let $<\sigma,\tau>: e \to c_0\times c_0$

$$
\begin{aligned}
[e,hom_c](<\sigma,\tau>) &= \{f: e\to c_1 \ / \ \rho_0 \circ f = <\sigma,\tau> \} \\
&= \{f: e\to c_1 \ / \ <DOM,COD> \circ f = <\sigma,\tau> \} \\
&= hom_{[e,c]}(\sigma,\tau);
\end{aligned}
$$

- on morphisms: let $<f,g> : <\sigma,\tau>\to <\gamma,\delta>$ in $[e,c^{op}\times c]$. $\forall h\in[e,hom_c](<\gamma,\delta>)$, i.e., for all $h: e\to c_1$ such that $<DOM,COD> \circ h = <\gamma,\delta>$, we have

$$
\begin{aligned}
[e,hom_c](<f,g>)(h) &= \rho_1 \circ <h, <f,g>>_0 \\
&= COMP \circ < p_2 \circ \Pi_2, COMP \circ (id \times_0 p_1) >_0 \circ <h, <f,g>>_0 \\
&= COMP \circ < g, COMP \circ <h, f>_0 >_0 \\
&= g \circledcirc h \circledcirc f . \ \blacklozenge
\end{aligned}
$$

The next definition finally externalizes the notion of morphism of presheaf that simply becomes a natural transformation. Proposition 7.4.10 states that the composition of an internal functor with a morphism of presheaf, given by the pulling back construction of definition 7.3.3, externalizes to the composition of the two associated external functors.

7.4.9 Definition *Let η be a morphism of presheaves from $X = (X, \rho_0 ,\rho_1)$ to $Y = (Y, \sigma_0,\sigma_1)$, where X an Y are internal presheaves on c. The natural transformation $[e,\eta]: [e,X]\to[e,Y]$ (where $[e,X], [e,Y]: [e,c]^{op}\to$ **Set**) is defined in the following way: $\forall\gamma\in[e,c]$, $\forall f\in[e,X](\gamma)$, $[e,\eta](\gamma)(f) = \eta \circ f$ (note that $[e,\eta](\gamma)(f) \in[e,Y](\gamma)$, since $\sigma_0 \circ \eta \circ f = \rho_0 \circ f = \gamma$).*

$[e,\eta]$ is indeed a natural transformation, since, $\forall g: \tau\to\gamma$ in $[e,c]$, $\forall f\in[e,X](\gamma)$

$$
\begin{aligned}
[e,Y](g) ([e,\eta](\gamma)(f)) &= [e,Y](g) (\eta \circ f) && \text{by def. of } [e,\eta] \\
&= \sigma_1 \circ <\eta \circ f, g>_0 && \text{by def. of } [e,Y](g) \\
&= \sigma_1 \circ \eta\times_0 id \circ <f, g>_0 && \\
&= \eta \circ \rho_1 \circ <f, g>_0 && \text{by the “naturality” of } \eta \\
&= \eta \circ ([e,X](g)(f)) && \text{by def. of } [e,X](g) \\
&= [e,\eta](\tau) ([e,X](g)(f)) && \text{by def. of } [e,\eta]
\end{aligned}
$$

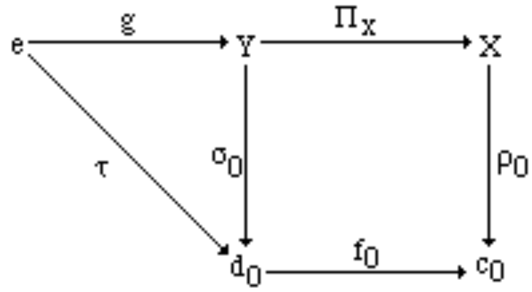
7.4.10 Proposition *Let $F: d\to c$ be an internal functor, $X = (X, \rho_0, \rho_1)$ an internal presheaf on c, and $F^*(X) = (Y, \sigma_0, \sigma_1)$. For every object e of E, the functors $[e,F^*(X)]$ and $[e,X]\circ[e,F]^{op}: [e,d]^{op}\to$* ***Set*** *are naturally isomorphic. The isomorphism is*

$$\eta_\tau = \lambda g.\ \Pi_X \circ g\ :\ [e,F^*(X)](\tau) \to [e,X]([e,F]^{op}(\tau))$$

$$\eta_\tau^{-1} = \lambda h.<\tau, h>_0 : [e,X]([e,F]^{op}(\tau)) \to [e,F^*(X)](\tau)$$

Proof Let us check first that η_τ and η_τ^{-1} have the correct types.

By definition $[e,F^*(X)](\tau) = \{g\in E[e,Y] \ /\ \sigma_0 \circ g = \tau\}$. Let $g\in[e,F^*(X)](\tau)$. Then the following diagram commutes:



Thus $\Pi_X \circ g \in [e,X](f_0 \circ \tau) = [e,X]([e,F]^{op}(\tau))$

Conversely, let $h\in[e,X]([e,F]^{op}(\tau))$. Then the arrow $<\tau, h>_0: e\to Y$ is well defined, because $\rho_0 \circ h = f_0 \circ \tau$. By definition of σ_0, $\sigma_0 \circ <\tau, h>_0 = \tau$ which implies $<\tau, h>_0\in[e,F^*(X)](\tau)$.

We now prove the naturality of η_τ and η_τ^{-1}. Let $k: \gamma\to\tau$ in $[e,d]^{op}$; for every $g\in[e,F^*(X)](\tau)$

$[e,X]([e,F]^{op}(k))(\eta_\tau(g))$	$= [e,X](f_1 \circ k) (\Pi_X \circ g)$	by def. of $[e,F]^{op}$
	$= \rho_1\circ <\Pi_X \circ g, f_1 \circ k >_0$	by def. of $[e,X]$
	$= \rho_1\circ \Pi_X\times_0 f_1 \circ < g, k >_0$	
	$= \Pi_X \circ \sigma_1 \circ < g, k >_0$	by def. of ρ_1
	$= \Pi_X \circ ([e,F^*(X)](k) (g))$	by def. of $[e,F^*(X)]$
	$= \eta_\gamma([e,F^*(X)](k) (g))$	by def. of η

Conversely, for every $k: \gamma\to\tau$ in $[e,d]^{op}$ and every $h\in[e,X]([e,F]^{op}(\tau))$:

$$[e,F^*(X)](k) (\eta_\tau^{-1}(h)) = \sigma_1 \circ < \eta_\tau^{-1}(h) , k >_0 = \sigma_1 \circ < <\tau, h >_0 , k >_0$$

Thus: $\Pi_X \circ ([e,F^*(X)](k) (\eta_\tau^{-1}(h))) = \Pi_X \circ \sigma_1 \circ < <\tau, h >_0 , k >_0$

$= \Pi_X \circ \sigma_1 \circ < <\tau, h >_0 , k >_0$

$= \rho_1\circ \Pi_X\times_0 f_1\circ < <\tau, h >_0 , k >_0$

$= \rho_1\circ < \Pi_X \circ <\tau, h >_0, f_1\circ k >_0$

$= \rho_1\circ < h, f_1 \circ k >_0$

and $\sigma_0 \circ ([e,F^*(X)](k) (\eta_\tau^{-1}(h))) = \sigma_0 \circ \sigma_1 \circ < <\tau, h >_0 , k >_0$

$= DOM \circ \Pi_2 \circ < <\tau, h >_0 , k >_0$

$= DOM \circ k$

$= \gamma$

And since $f = <\sigma_0 \circ f, \Pi_X \circ f >$, then for every $f : e\to Y$,

$[e,F^*(X)](k)(\eta_\tau{}^{-1}(h)) = <\gamma, \rho_1 \circ <h, f_1 \circ k>_0>_0$
$= <\gamma, [e,X]([e,F]^{op}(k))(h)>_0$
$= \eta_\gamma{}^{-1}([e,X]([e,F]^{op}(k))(h))$. ♦

7.4.11 Proposition *Let $<F, G, \phi> : c \to d$ be an internal adjunction. For every e in E, define $\Theta_e = \eta' \circ [e,\phi] \circ \eta^{-1}$, where*

$$\eta : [e,(F\times Id_{d^{op}})^*(hom_d)] \to [e, hom_d] \circ [e,F^{op}\times Id]$$
$$\eta' : [e,(Id_c\times G^{op})^*(hom_c)] \to [e, hom_c] \circ [e,Id\times G^{op}]$$

are the isomorphisms of proposition 7.4.10 .

Then $<[\_,F], [\_,G], \Theta> : [\_,c] \to [\_,d]$ is an E-indexed adjunction.

Proof For every object e of E, we have

$$\begin{aligned} hom_{[e,d]}[[e,F](\_), \_] &= [e, hom_d] \circ [e,F^{op}\times Id] \\ &\cong [e,(F\times Id_{d^{op}})^*(hom_d)] && \text{via } \eta^{-1} \\ &\cong [e,(Id_c\times G^{op})^*(hom_c)] && \text{via } [e,\phi] \\ &\cong [e, hom_c] \circ [e,Id\times G^{op}] && \text{via } \eta' \\ &= hom_{[e,c]}[\_, E_{e,G}(\_)]. \end{aligned}$$

Moreover, the previous adjunction is “natural in e,” that is,

$$\forall f\in E[e',e] \quad \Theta_{e'} \circ [\_,d](f) = [\_,c](f) \circ \Theta_e .$$

More explicitly, we must check that, for every $f\in E[e',e]$, σ object of $[e,c]$, τ object of $[e,d]$, and $g: (f_0 \circ \sigma) \to \tau$ in $[e,d]$, one has

$$\Theta_{e'} <\sigma \circ f, \tau \circ f> ([\_,d](f))(g) = ([\_,c](f))\, \Theta_e<\sigma,\tau>(g)$$

We have

$$\begin{aligned} \Theta_{e'} <\sigma \circ f, \tau \circ f> ([\_,d](f))(g) &= \Theta_{e'} <\sigma \circ f, \tau \circ f> (g \circ f) && \text{by def. of } [\_,d] \\ &= \Theta_{e'} <\sigma \circ f, \tau \circ f> (g \circ f) \\ &= \Pi_X \circ \phi \circ <<\sigma \circ f, \tau \circ f>, g \circ f>_0 && \text{by def. of } \Theta_{e'} \\ &= \Pi_X \circ \phi \circ <<\sigma,\tau>, g>_0 \circ f \\ &= (\Theta_e<\sigma,\tau>(g)) \circ f && \text{by def. of } \Theta_e \\ &= ([\_,c](f))\, \Theta_e<\sigma,\tau>(g) && \text{by def of } [\_,c]. \end{aligned}$$ ♦

7.4.12 Exercise Prove that if $<F, G, \phi> : c \to d$ is an internal adjunction, and Unit and Counit are the arrows in theorem 7.3.7, than for every object $\sigma: e \to d_0$ in $[e,d]$, Unit $\circ\ \sigma$, Counit $\circ\ \sigma$ are respectively unit and counit for σ in the associated external adjunction $<[e,F], [e,G], \Theta_e>: [e,c] \to [e,d]$.

7.5 Internalization

In this section we show how to translate (small) E-indexed notions to internal ones in the topos of presheaves $E^{op}\to$**Set**.

7.5.1 Definition *Let A: $E^{op}\to$**Cat** be an E-indexed category, where all the indexed categories are small. The internal category $\underline{A} = (\underline{A}_0, \underline{A}_1, \underline{DOM}, \underline{COD}, \underline{COMP}, \underline{ID})\in Cat(E^{op}\to$**Set**$)$ is defined as follows: for all objects e, e' and arrows f: e'→e in E,*

- *$\underline{A}_0: E^{op}\to$**Set** is the functor defined by*

$$\underline{A}_0(e) = Ob_{A(e)}$$
$$\underline{A}_0(f) = A(f)_{ob} : Ob_{A(e)}\to Ob_{A(e')}$$

- *$\underline{A}_1: E^{op}\to$**Set** is the functor defined by*

$$\underline{A}_1(e) = Mor_{A(e)}$$
$$\underline{A}_1(f) = A(f)_{mor} : Mor_{A(e)}\to Mor_{A(e')}$$

- *$\underline{DOM}: A_1\to A_0$ is the natural transformation whose components are the domain maps in the local categories, i.e., for $e\in Ob_E$, $\underline{DOM}_e: Mor_{A(e)}\to Ob_{A(e)}$ is defined by $\underline{DOM}_e(h:\sigma\to\tau) = \sigma$.*
- *$\underline{COD}$, $\underline{ID}$ and $\underline{COMP}$ are defined analogously, "fiberwise".*

The claimed naturality for $\underline{DOM}$, $\underline{COD}$, $\underline{ID}$, $\underline{COMP}$ is immediate, since A is a functor. For instance, let $f\in E[e',e]$ and $h\in A(e)[\sigma,\tau]$; then $\underline{DOM}_{e'}(A(f)_{mor}(h)) = A(f)_{ob} \circ \underline{DOM}_e(h)$. The reader can check the other cases as an exercise.

7.5.2 Definition *Let A, B be two E-indexed categories, and let H: A→B be an E-indexed functor.The associated **internal functor** $\underline{H} = (\underline{H}_0,\underline{H}_1): \underline{A}\to\underline{B}$ in $E^{op}\to$**Set**, is defined in the following way:*

- *$\underline{H}_0: \underline{A}_0\to\underline{B}_0$ is the natural transformation given by $\underline{H}_0(e) = H(e)_{ob}$*
- *$\underline{H}_1: \underline{A}_1\to\underline{B}_1$ is the natural transformation given by $\underline{H}_1(e) = H(e)_{mor}$*

The naturality of $\underline{H}_0$ and $\underline{H}_1$ is an immediate consequence of the "naturality" of H: A→B, that is $H(s) \circ A(f) = B(f) \circ H(s')$. The equations in definition 7.2.3 easily follow from the fact that for every e, H(e) is a functor.

7.5.3 Definition *Let H: A→B, K: A→B be two E-indexed functors, and let τ: H→K be an E-indexed natural transformation. Then the associated internal natural transformation $\underline{\tau}: \underline{H}\to\underline{K}$ in $E^{op}\to$**Set** is the natural transformation $\underline{\tau}: \underline{A}_0\to\underline{B}_1$ such that, for any e in E, and any a in A(e), $\underline{\tau}_e(a) = \tau(e)_a$.*

Recall that τ: H→K consists of a natural transformation τ(e): H(s)→K(e) for any object e of E, such that, for any f: e→e' in E, and any object a in A(e'), $\tau(e)_{A(f)(a)} = B(f)(\tau(e')_a)$.
As a consequence, $\underline{\tau}_e(A_0(f)(a)) = B_1(f)(\underline{\tau}_{e'}(a))$, which gives the naturality of $\underline{\tau}$.

7.5.4 Proposition *Let A, B be E-indexed categories, H: A→B, K: B→A be E-indexed functors, and <H, K, ϕ> : A→B be an E-indexed adjunction. Then <$\underline{H}$, $\underline{K}$, $\underline{\phi}$ = ϕ> : $\underline{A}$→$\underline{B}$ is an internal adjunction in* E^{op}*→**Set**.*
Proof Exercise.

The picture is finally completed by the following result, which shows that by applying the externalization process of section 7.4 to an internal category $\underline{A}$, derived from an E-indexed category A, we obtain an indexed category equivalent to A. However, when we externalize $\underline{A}$, we do not want a category indexed over *all* functors from E into **Set**. Since we are interested in a category indexed over E, we must externalize only with respect to a full subcategory of E^{op}→**Set** equivalent to E. The obvious choice is to consider the image Y(E) of E under the Yoneda embedding Y(e) = E[_, e] (recall that Y(E) is a full subcategory of E^{op}→**Set**). We will then obtain an indexed category $\underline{A}^{\#}$: E^{op}→**Cat**. Recall though that the "internalization" can take place only if the indexed category takes small categories as values, while internal categoies do not need to live in small ambient categories. Thus, the circle is closed by the following theorem, provided that the assumption is made that E is small.

7.5.5 Theorem *Let A: E^{op}→**Cat** be an E-indexed category, with E small, and let $\underline{A}$∈Cat(E^{op}→**Set**) be its associated internal category. Then the indexed categories $\underline{A}^{\#}$ = [Y(_), $\underline{A}$] : E^{op}→**Cat** and A are equivalent.*
Proof Let e∈Ob_E. Then $\underline{A}^{\#}$(e) = [Y(e), $\underline{A}$] is, by definition, the category with

Objects: Nat[E[_,e], $\underline{A}_0$]
Morphisms: g: σ→τ in [Y(e), $\underline{A}$] iff
g∈Nat[E[_,e], $\underline{A}_1$] , $\underline{DOM}\circ g = \sigma$, $\underline{COD}\circ g = \tau$

Now let f∈E[e',e]; by definition one has

$\underline{A}^{\#}$(f) = [Y(f), $\underline{A}$] : $\underline{A}^{\#}$(e)→ $\underline{A}^{\#}$(e')
$\underline{A}^{\#}(f)(\sigma) = \sigma \circ Y(f)$ for σ∈Nat[E[_,e], $\underline{A}_0$]
$\underline{A}^{\#}(f)(g) = g \circ Y(f)$ for g∈Nat[E[_,e], $\underline{A}_1$]

The natural isomorphism between A and $\underline{A}^{\#}$ is given by the Yoneda lemma: for every e in E, we have natural isomorphisms Ψ_0(e): Nat[E[_,e], $\underline{A}_0$]→$\underline{A}_0$(e) and Ψ_1(e): Nat[E[_,e], $\underline{A}_0$]→$\underline{A}_1$(e). Ψ_0 and Ψ_1 define the components on objects and morphisms of an indexed functor Ψ(e): $\underline{A}^{\#}$(e)→A(e). Explicitly,

$\Psi(e)(\sigma) = \sigma_e(id_e)$ for σ∈Nat[E[_,e], $\underline{A}_0$]
$\Psi(e)(g) = g_e(id_e)$ for g∈Nat[E[_,e], $\underline{A}_1$].

Then the due diagrams commute, by the usual Yoneda argument.

Our final result shows that, by following the other path (from internal to internal, via external), one obtains equivalent categories:

7.5.6 Theorem *Let $c \in Cat(E)$ be an internal category, $C = [\_,c]: E^{op} \to$* ***Cat*** *be as in definition 7.4.2, and $Y: E \to Y(E)$ be the Yoneda embedding. Then $\underline{C} \in Cat(Y(E))$.*

Proof Let $c = (c_0, c_1, DOM, COD, COMP, ID)$; note first that, by definition of C, for the internal category $\underline{C} = (d_0, d_1, DOM', COD', COMP', ID') \in Cat(E^{op} \to \mathbf{Set})$ we have

$d_0 = E[\_, c_0] = Y(c_0)$

$d_1 = E[\_, c_1] = Y(c_1)$

and hence $\underline{C} \in Cat(Y(E))$, since Y is full. That $\underline{C}$ is an internal category, follows by the fact that Y preserves pullbacks. ♦

Appendix

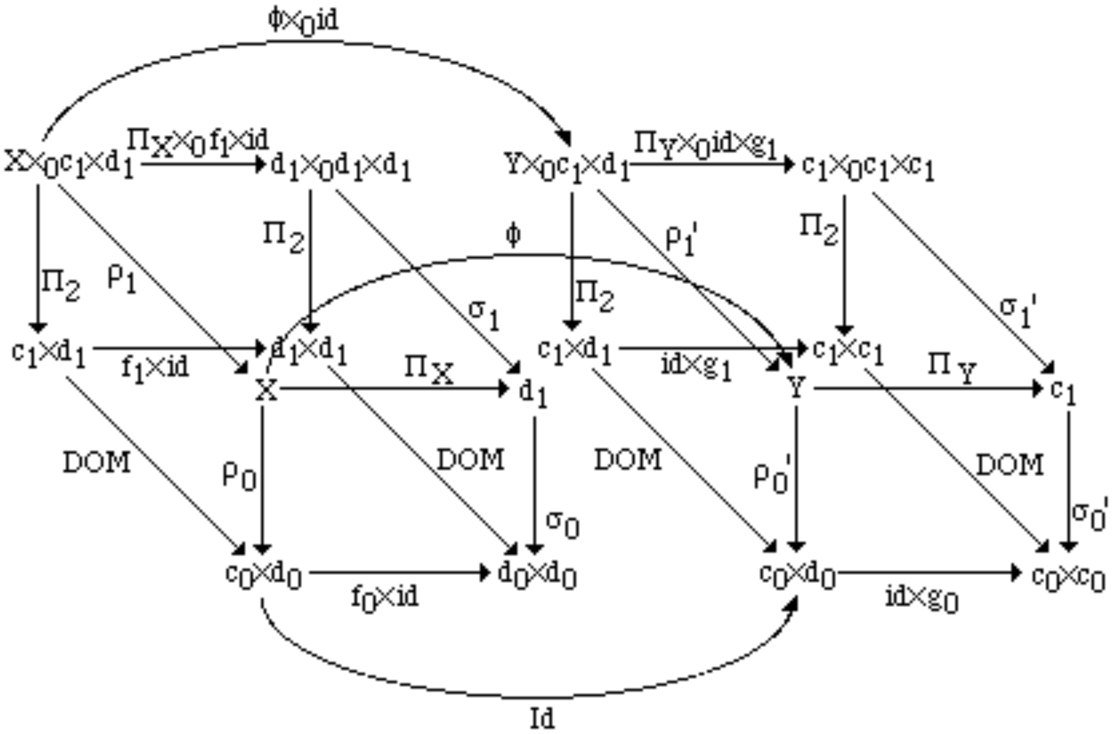
We now study in more details the notions of internal adjunction and internal CCC. The details are rather complex and this appendix may be skipped at first reading.

By definition, an internal adjunction $< F, G, \phi > : c \to d$ is given by two internal functors $F: c \to d$, $G: d \to c$, and an isomorphism

$$\phi : (F \times Id_d{}^{op})^*(hom_d) \to (Id_c \times G^{op})^*(hom_c)$$

between presheaves on $c \times d^{op}$.

Graphically, the notion of internal adjunction is represented by the following complex diagram:



where (d_1,σ_0,σ_1) is the internal hom-functor of d, and $(c_1,\sigma_0',\sigma_1')$ is the internal hom-functor of c. In particular,

$$\sigma_0 = \langle DOM,COD\rangle : d_1 \to d_0\times d_0 \text{ and}$$
$$\sigma_0' = \langle DOM,COD\rangle : c_1 \to c_0\times c_0$$

respectively represent d_1 and c_1 as indexed collections of morphisms over $d_0\times d_0$ and $c_0\times c_0$. The formal definition of σ_1 and σ_1' is:

$$\sigma_1 = COMP \circ \langle p_2 \circ \Pi_2, COMP \circ (id \times_0 p_1) \rangle_0 : \ d_1\times_0(d_1\times d_1) \to d_1$$
$$\sigma_1' = COMP \circ \langle p_2 \circ \Pi_2, COMP \circ (id \times_0 p_1) \rangle_0 : \ c_1\times_0(c_1\times c_1) \to c_1$$

More intuitively, they are both described by the lambda term $\lambda fgh.\ h \circ f \circ g$ (recall that $hom[f,g](h) = h \circ f \circ g$).

Note also that $DOM: c_1\times d_1\to c_0\times d_0 = DOM_c\times COD_d$ because we are working in $c\times d^{op}$.

X and Y are respectively the pullbacks of

$$\sigma_0 = \langle DOM,COD\rangle : d_1 \to d_0\times d_0\ ,\ f_0\times id : c_0\times d_0 \to d_0\times d_0 \text{ and}$$
$$\sigma_0' = \langle DOM,COD\rangle : c_1 \to c_0\times c_0\ ,\ id\times g_0 : c_0\times d_0 \to d_0\times d_0$$

Thus, informally,

$$X = \{ (a,b,h) \in c_0\times d_0\times d_1 \mid h : f_0(a) \to b \} = d[f_0(a), b]$$
$$Y = \{ (a,b,k) \in c_0\times d_0\times c_1 \mid k : a \to g_0(b) \} = c[a, g_0(b)].$$

$\phi : X \to Y$ is the natural isomorphism of the adjunction.

ϕ works on triples of the kind $(a,b,h)\in c_0\times d_0\times d_1$ where $h: f_0(a)\to b$. The first two components a and b are the indexes of the natural trasformation: since $\rho_0'\circ \phi = \rho_0$, these indexes are left unchanged by ϕ, and an "external-like" writing for $\phi(a,b,h)$ would be $\phi_{a,b}(h)$. At the external level, it is common practice to omit these indexes; the formal complexity of the internal theory is mostly due to the necessity of coping with these details.

The naturality of ϕ is expressed by the property,

$$(\dagger) \qquad \phi \circ \rho_1 = \rho_1'\circ \phi\times_0 id\ .$$

Still using our informal notation, by (†), for all (a,b,h) in X, k in c_1 and l in d_1, such that :

$$cod(k) = a \quad (\text{ that implies } cod(f_0(k)) = f_0(a) = dom(h))$$
$$dom(l) = b = cod(h)$$
$$cod(l) = b'$$

we have

$$(*) \qquad \phi_{a',b'}(l \circ h \circ f_1(k)) = g_1(l) \circ \phi_{a,b}(h) \circ k,$$

that is the familiar way the naturality of ϕ is expressed at the external level. Let us show, in this informal notation, that (†) implies (*)

$$\phi_{a',b'}(l \circ h \circ f_1(k)) =$$
$$= (\Pi_Y \circ \phi)(a', b' , l \circ h \circ f_1(k))$$

$= (\Pi_Y \circ \phi)(\ a', b'\ , \sigma_1(h, f_1(k), l\ )\ )$	by def. of σ_1
$= (\Pi_Y \circ \phi \circ \rho_1)\ ((a,b,h), k, l\ )$	by the diagram for the adjunction
$= (\Pi_Y \circ \rho_1' \circ \phi \times_0 id)\ (\ (a,b,h), k, l\ )$	by (†)
$= (\sigma_1' \circ \Pi_Y \times_0 (id \times g_1) \circ \phi \times_0 id)\ (\ (a,b,h), k, l\ )$	by the diagram for the adjunction
$= \sigma_1'(\ (\Pi_Y \circ \phi)(a,b,h), k, g_1(\ l\ )\ )$	
$= \sigma_1'(\ \phi_{a,b}(h), k,\ g_1(\ l\ )\ )$	
$= g_1(\ l\ ) \circ \phi_{a,b}(h) \circ k$	by def. of σ_1'.

Given an adjunction $< F, G, \phi > : C \to D$, the arrows $\phi_{a,F(a)}(id_{F(a)})$ and $\phi_{G(b),b}{}^{-1}(id_{G(b)})$ are respectively called Unit and Counit of the adjunction (for a and b). Units and Counits fully specify the behaviour of ϕ and ϕ^{-1} since:

$$\phi(\ l\ ) = \phi(\ l \circ id\ ) = g_1(\ l\ ) \circ \phi(id) = g_1(\ l\ ) \circ \text{Unit}$$

$$\phi^{-1}(k) = \phi^{-1}(id \circ k) = \phi^{-1}(id) \circ F(k) = \text{Counit} \circ F(k)\ .$$

These properties allow to give at the external level the well-known equational characterization of the notion of adjunction. In particular, the definition of Cartesian closed category based on the counits of the adjunctions, plays a central role in the semantic investigation of the lambda calculus, since it provides the underlying applicative structure needed for the interpretation. Remember that the counits of the adjunctions defining products and exponents are respectively the projections associated with the products and the evaluation functions associated with the function spaces.

Now we show how to mimic the same work at the internal level.

7.A.1 Definition *Let $< F, G, \phi > : c \to d$ be an internal adjunction from c to d. Define then:*

ID$_F$ $= <<id,f_0>, ID \circ f_0 >_0 : c_0 \to X;$

ID$_G$ $= <<g_0,id>, ID \circ g_0 >_0 : d_0 \to Y;$

Unit $= \Pi_Y \circ \phi \circ ID_F : c_0 \to c_1;$

Counit $= \Pi_X \circ \phi^{-1} \circ ID_G : d_0 \to d_1.$

Where X and Y are as in the diagram for the definition of adjunction.

Note that ID_F takes an element a in c_0 and gives the associated identity $id_{F(a)}$ as an element in X. The definition of Unit, is then clear. As one expects, Unit is an internal natural transformation from I = (id,id) to $G \circ F$, and Counit : $d_0 \to d_1$ is an internal natural transformation from $F \circ G$ to I = (id,id). The proof is left as an exercise for the reader.

It is now not difficult to prove that every internal adjunction $< F, G, \phi > : c \to d$ is fully determined by the following data:

the functor $G: d \to c$;

an arrow $f_0: c_0 \to d_0$;

an arrow Unit: $c_0 \to c_1$ such that $DOM \circ \text{Unit} = id$, $COD \circ \text{Unit} = g_0 \circ f_0$;

an arrow ϕ^{-1}: Y→X, where X and Y are respectively the pullbacks of

<DOM,COD> : $d_1 \to d_0 \times d_0$, $f_0 \times id$: $c_0 \times d_0 \to d_0 \times d_0$, and

<DOM,COD> : $c_1 \to c_0 \times c_0$, $id \times g_0$: $c_0 \times d_0 \to c_0 \times c_0$;

and, moreover, the previous functions satisfy the following equations:

a. $< \rho_0, COMP \circ < g_1 \circ \Pi_X, Unit \circ p_1 \circ \rho_0 >_0 >_0 \circ \phi^{-1} = id_Y$;

b. $\phi^{-1} \circ < \rho_0, COMP \circ < g_1 \circ \Pi_X, Unit \circ p_1 \circ \rho_0 >_0 >_0 = id_X$.

Indeed the arrow f_0: $c_0 \to d_0$ can be extended to a functor $F = (f_0, f_1)$: c→d by

$f_1 = \Pi_X \circ \phi^{-1} \circ << DOM, f_0 \circ COD >, COMP \circ < Unit \circ COD, id >_0 >_0: c_1 \to d_1$.

The inverse of ϕ^{-1} is

$\phi = < \rho_0, COMP \circ < g_1 \circ \Pi_X, Unit \circ p_1 \circ \rho_0 >_0 >_0$.

Note that, by (a) and (b), ϕ and ϕ^{-1} define an isomorphism. The non trivial fact is to prove that they are morphisms of presheaves (i.e., to prove their naturality), but again the prof is a mere internal rewriting of the corresponding "external" result.

Dually, if we have the following data:

a functor F: c→d;

an arrow g_0: $d_0 \to c$;

an arrow Counit: $d_0 \to d_1$ such that $DOM \circ Counit = f_0 \circ g_0$, $COD \circ Counit = id$;

an arrow ϕ: X→Y, where X and Y are respectively the pullbacks of

<DOM,COD> : $d_1 \to d_0 \times d_0$, $f_0 \times id$: $c_0 \times d_0 \to d_0 \times d_0$, and

<DOM,COD> : $c_1 \to c_0 \times c_0$, $id \times g_0$: $c_0 \times d_0 \to c_0 \times c_0$;

and, moreover, the previous functions satisfy the following equations:

a. $< \rho_0', COMP \circ < Counit \circ p_2 \circ \rho_0', f_1 \circ \Pi_Y >_0 >_0 \circ \phi = id_X$,

b. $\phi \circ < \rho_0', COMP \circ < Counit \circ p_2 \circ \rho_0', f_1 \circ \Pi_Y >_0 >_0 = id_Y$,

then we define an adjunction $< F, G, \phi >$: c→d, in the following way.

The arrow g_0: $d_0 \to c_0$ can be extended to a functor $G = (g_0, g_1)$: c→d, by

$g_1 = \Pi_Y \circ \phi \circ << g_0 \circ DOM, COD >, COMP \circ < id, Counit \circ DOM >_0 >_0: d_1 \to c_1$.

The inverse of ϕ is

$\phi^{-1} = < \rho_0', COMP \circ < Counit \circ p_2 \circ \rho_0' , f_1 \circ \Pi_Y >_0 >_0 : Y \to X$.

We are now in a position to study internal Cartesian closed categories from an "equational" point of view. This work is needed to exploit the applicative structure underlying the notion of an internal CCC. Recall that an internal Cartesian closed category is a category c∈Cat(E) with three adjunctions

1. < O, T, $\mathbb{O}$ > : c→1, where 1 is the internal terminal category;
2. < Δ, x, ⟪,⟫ > : c→c×c, where Δ is the internal diagonal functor;
3. < x, [,] , $\mathbb{A}$ > : c→c, where this adjunction has parameters in c.

By the previous results, we can explicitate the three adjunctions of these definitions by means of their counits:

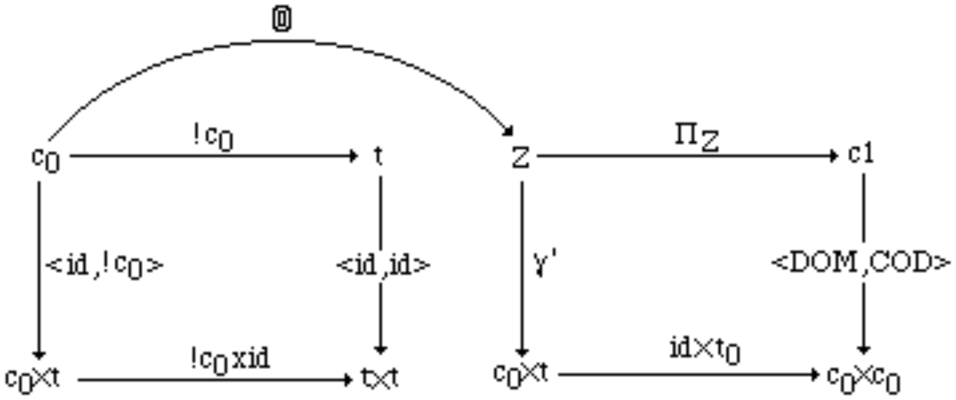
7.A.2 Definition *An* ***internal terminal object*** *in* $c \in Cat(E)$ *is specified by:*

an arrow $t_0: t \to c_0$;

an arrow $\mathbb{O}: c_0 \to Z$, *where Z is the pullback of*

$<DOM,COD> : c_1 \to c_0 \times c_0$,

$id \times t_0 : c_0 \times t \to c_0 \times c_0$;



and, moreover,

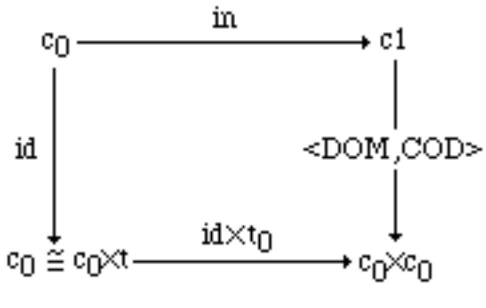
a. $\mathbb{O} \circ <\gamma',!Z>_0 = \mathbb{O} \circ p_1 \circ \gamma' = id_Z$;

b. $<\gamma',!Z>_0 \circ \mathbb{O} = p_1 \circ \gamma' \circ \mathbb{O} = id_{c_0}$;

where !Z is the unique morphism in E from Z to the terminal object t .

Intuitively $t_0: t \to c_0$ points to that element in c_0 that is the terminal object. Z is the subset of c_1 of all those morphisms that have the terminal object as target; Z must then be in a bijective relation $\mathbb{O}$ with c_0; $\mathbb{O}$ takes an object a in c_0 to the unique morphism $!_a$ in Z from a to the terminal object.

The previous diagram can be greatly simplified. As a matter of fact, it amounts to say that there is an arrow $t_0: t \to c_0$ such that the following diagram is a pullback (prove it as an exercise):



The arrow $in: c_0 \to c_1$ is the operation that takes every element a in c_0 to the unique arrow !a in c_1 whose target is the terminal object; in terms of the previous diagram, $in = \Pi_Z \circ \mathbb{O}$.

7.A.3 Definition *An internal category* c ***has products****, iff there exist*

an arrow $x_0: c_0 \times c_0 \to c_0$;

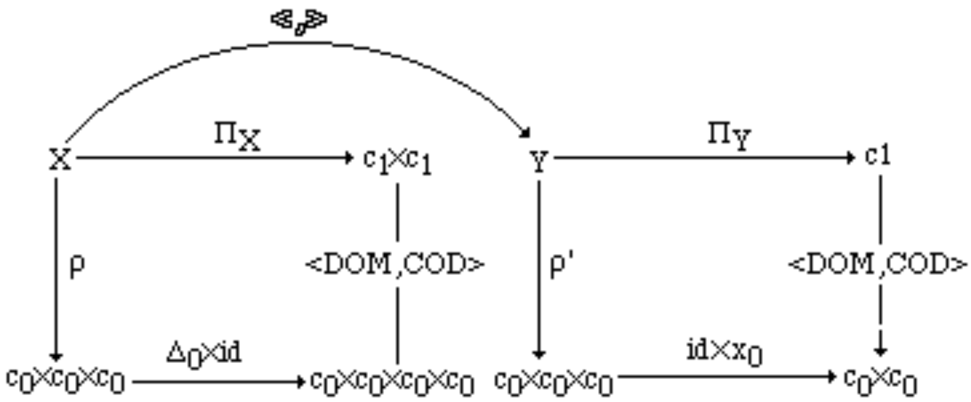
two arrows $FST: c_0 \times c_0 \to c_1$, $SND: c_0 \times c_0 \to c_1$ *such that*

$DOM \circ FST = DOM \circ SND = x_0$,

$COD \circ FST = p_1$; $COD \circ SND = p_2$,

(Notation: $FST_{a,b} = FST \circ <a,b>$; $SND_{a,b} = SND \circ <a,b>$*);*

an arrow $\ll,\gg : X \to Y$, *where* X *and* Y *are the pullbacks in the following diagram* $(\Delta_0 = <id,id>)$:



and, moreover,

c0. $\rho' \circ \ll,\gg = \rho$;

c1. $(FST \circ p_2 \circ \rho) \circledcirc (\Pi_Y \circ \ll,\gg) = p_1 \circ \Pi_X$;

c2. $(SND \circ p_2 \circ \rho) \circledcirc (\Pi_Y \circ \ll,\gg) = p_2 \circ \Pi_X$;

d. $\ll,\gg \circ < \rho', < (FST \circ p_2 \circ \rho') \circledcirc \Pi_Y , (SND \circ p_2 \circ \rho') \circledcirc \Pi_Y > >_0 = id_Y$,

where $f \circledcirc g = COMP \circ < f, g >_0$.

7.A.4 Definition *An internal category is* ***Cartesian*** *iff it has a terminal object and products.*

As the definition $f \times g = <f \circ p_1, g \circ p_2>$ extends $\times$ to a functor from C×C to C for any Cartesian C, also the internal x_0 can also be extended to morphisms.

7.A.5 Proposition *Let* $x_1 : c_1 \times c_1 \to c_1$ *be defined by the following:*

$x_1 = \Pi_Y \circ \ll,\gg \circ < <x_0 \circ DOM_{c\times c}, COD_{c\times c} >, <id \circledcirc (FST \circ DOM_{c\times c}), id \circledcirc (SND \circ DOM_{c\times c}) >_0$

where as above, $f \circledcirc g = COMP \circ <f,g>_0$ *. Then* $x = (x_0, x_1): c\times c \to c$ *is an internal functor.*

Proof: Exercise.

Note that, if $f,g: e \to c_1$, $DOM \circ f = a$, $COD \circ f = c$, $DOM \circ g = b$, $COD \circ g = d$, then: $x_1 \circ <f,g> = \Pi_Y \circ \ll,\gg \circ < < x_0 \circ <a,b>,<c,d>>,< f \circledcirc FST_{a,b} , g \circledcirc SND_{a,b} > >_0$.

7.A.6 Definition *An internal Cartesian category has* ***exponents*** *iff there exist :*

an arrow $[,]_0: c_0 \times c_0 \to c_0$;

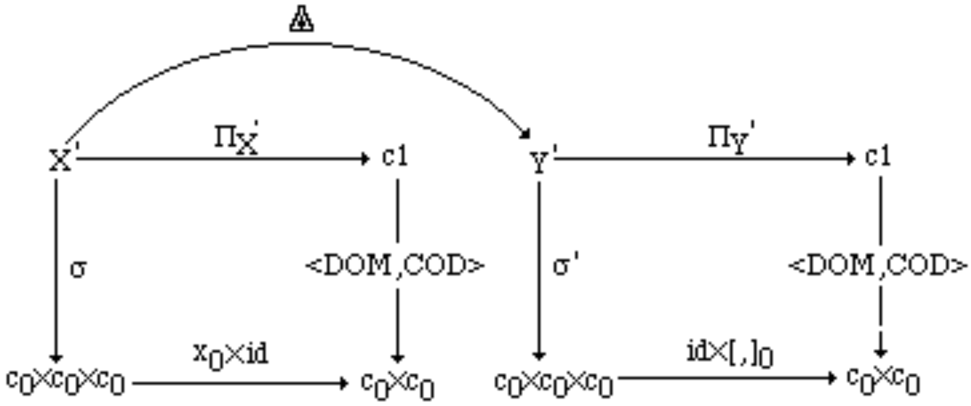
an arrow $EVAL: c_0 \times c_0 \to c_1$ *such that*

$DOM \circ EVAL = \times_0 \circ <[,]_0, p_1>$,

$COD \circ EVAL = p_2$,

(Notation: $EVAL_{a,b} = EVAL \circ <a,b>$ *);*

an arrow $\mathbb{A}: X' \to Y'$, *where* X' *and* Y' *are the pullbacks in the following diagram:*



and, moreover,

e0. $\sigma' \circ \mathbb{A} = \sigma$ *(to within the isomorphism* $(a\times b)\times c \cong a\times(b\times c)$ *);*

e1. $(eval \circ p_1 \circ \sigma) \circledcirc (x_1 \circ < \Pi_{Y'} \circ \mathbb{A}, ID \circ p_2 \circ p_1 \circ \sigma>) = \Pi_{X'}$;

f. $\mathbb{A} \circ < \sigma', (eval \circ p_2 \circ \sigma') \circledcirc (x_1 \circ < \Pi_{Y'}, ID \circ p_1 \circ p_2 \circ \sigma'>) >_0 = id_{Y'}$,

where $f \circledcirc g = COMP \circ < f, g >_0$, *and* x_1 *is the morphism in proposition A.5.*

7.A.7 Definition *An* ***internal Cartesian closed category*** *is an internal Cartesian category with exponents.*

References The introduction of indexed notions in Category Theory has been a slow process, which started to develop into a detailed theory around the beginning of the 1970s, with the independent work of F. W. Lawvere, J. Penon, J. Bénabou. The Theory of Indexed Category owes much of its actual settlement to R. Paré and D. Schumacher (1978), and to their introductory book, written in collaboration with P. T. Johnstone, R. D. Rosebrugh, R. J. Wood and G. C. Wraith.

The first significant study of the Theory of Internal categories is due to R. Diaconescu. Further developments were made by J. Bénabou, though, most of the time, they never appeared as published works. Notions of Internal Category Theory can be found in many books of topos theory, for instance, in those of Johnstone (1977) and Barr and Wells (1985).

Our development of the arguments in this chapter has been essentially inspired by Paré and Schumacher (1978), Johnstone (1977), and some private communications of E. Moggi. The definition of the internal category in definition 7.5.1 has been pointed out to us by B. Jacobs.