Chapter 10

RECURSIVE DOMAIN EQUATIONS

One of the early applications of Category Theory to computer science was the solution of recursive domain equations. This kind of equation is typical of every language that allows an explicit or implicit form of self-application of its data types (such as a recursive procedure call, say).

For example, by theorem 9.5.10 we already know that in order to give semantics to the pure λ-calculus we need a domain isomorphic to its own function space; moreover, if we are interested in computing over a fixed domain A of atoms, we need a solution to the equation

$$(*)\quad X \cong A+(X\to X).$$

In general, recursive specification of domains can be seen as a particular case of recursive definition of data types that is an even more important topic in computer science. For example every programmer is used to considering the data type of all the lists of objects of type A as a solution to the following recursive equation:

$$\text{A\_List} = \text{Nil} + A\times(\text{A\_list})$$

where Nil is a one-element data type.

In many respects, the general and unified theory of this kind of equation, mainly developed in the framework of Category Theory, has provided the base for the elimination, in most modern languages, of many unreasonable restrictions in the definition of recursive data types that existed in older languages. Note that these restrictions were not always motivated by implementation problems, but often by a real misunderstanding of the semantics of recursive definitions.

The first mathematical difficulty in giving a meaning to recursive specifications of data types is that they do not always have a set-theoretic solution. For example, equation (*) above has no solution in **Set** by obvious cardinality reasons: the arrow-domain $A\to B$ cannot be interpreted as the collection of all functions between A and B. The natural choice is to consider categories other than **Set**, with fewer morphisms but sufficiently many as to include all “computable functions” over the intended data types. The relevance of morphisms suggests why the categorial framework arises so naturally in this context. Note that the intended category **C** must be still Cartesian closed, in order to give the correct interpretation to the function space constuction. Once **C** is fixed, the idea for solving recursive domain equations is to use some sort of *fixed point technique*. To be more specific, in the categorical approach, the general form of equations such as (*) above, looks like

$$(**)\quad X \cong F(X)$$

where X ranges over the object of a category **C** and $F: \mathbf{C}\to\mathbf{C}$ is a covariant endofunctor of the category. If **C** is ω-cocomplete, **C** has an initial object, and F is ω-continuous, then theorem 6.5.2

gives a solution to (**). The main problem is that an obvious definition does not always exist for such an F, as we are going to show in the next section.

10.1 The Problem of Contravariant Functors

Consider again the equation

$$(*) \quad X \cong A+(X\to X)$$

in the introduction. In order to apply theorem 6.5.2, we need to work in a ω-cocomplete category **C**. Moreover, the category must be Cartesian closed and have coproducts, so that we can give the expected interpretation to the operators → and +, which appear in (*).

Such a category is not difficult to find: an example is the category **CPOS** of c.p.o.'s with a least (bottom) element and strict (bottom-preserving) continuous functions for morphisms. The next step is to define a covariant functor $F: \mathbf{C}\to\mathbf{C}$ such that, for any object X of **C**, $F(X) = A+(X\to X)$. The first idea would be to express F as a composition of the functors of sum and exponentiation associated to the closure properties of **C**. For example, if $A+\_: \mathbf{C}\to\mathbf{C}$ is the functor which takes B to A+B, and f to id_A+f, exp is the exponentiation functor, and Δ is the diagonal functor, we could be tempted to define

$$F = (A+\_) \circ \exp \circ \Delta$$

Unfortunately, this is not possible since exp is contravariant in the first component and cannot be composed with the diagonal function; in other words, since $\exp: \mathbf{C}^{op}\times\mathbf{C}\to\mathbf{C}$, and $\Delta: \mathbf{C}\to\mathbf{C}\times\mathbf{C}$, the previous equation is ill typed.

We would like to have a way of transforming every functor F in an associated functor F^* with the same behavior on objects, but covariant on morphisms, in all its components.

Unfortunately this is not possible in general, but still there is a very simple way, as we will see, to turn a contravariant endofunctor $F: \mathbf{C}\to\mathbf{C}$ into a covariant endofunctor F^* in an associated category $\mathbf{C}^*$, which has the same objects of **C**, and such that isomorphisms in $\mathbf{C}^*$ give isomorphisms in **C**.

Then, if $\mathbf{C}^*$ is ω-cocomplete and has a terminal object, and F^* is ω-continuous, we can apply theorem 6.5.2, find a solution in $\mathbf{C}^*$, and then derive a solution in **C.** Note that the category $\mathbf{C}^*$ does not need to be cartesian closed, neither it must be closed under coproducts, since we already know how to give a meaning to the constuctions of new objects in **C**.

We shall define $\mathbf{C}^*$ as a suitable subcategory of the following category $\mathbf{C}^{+-}$.

10.1.1 Definition *Given a category* **C**, *the category* $\mathbf{C}^{+-}$ *has the same object of* **C** *and*

$$f \in \mathbf{C}^{+-}[A, B] \quad iff \quad f = (f^+,f^-) \ \ with \ \ f^+\in\mathbf{C}[A, B] \ \ and \ \ f^-\in\mathbf{C}[B, A].$$

Composition is defined by $(f^+,f^-) \circ (g^+,g^-) = (f^+ \circ g^+, g^- \circ f^-)$.

An intuitive way to regard the category $\mathbf{C}^{+-}$ is the following. Consider the objects as data types; there is a morphism from A to B if and only if a pair of "coercions" is given in order to go from one to the other. Note also that two objects are isomorphic in $\mathbf{C}^{+-}$ iff they are isomorphic in **C.**

We next show how to define covariant functors on $\mathbf{C}^{+-}$ from arbitrary functors on **C**. For simplicity, we reduce the definition to the case of a bifunctor F covariant in the first component and contravariant in the second.

10.1.2 Definition *Given a category* $\mathbf{C}$*, and a functor* $F: \mathbf{C}\mathrm{x}\mathbf{C} \to \mathbf{C}$ *contravariant in the first component and covariant in the second, the covariant functor* $F^{+-}: \mathbf{C}^{+-}\mathrm{x}\mathbf{C}^{+-} \to \mathbf{C}^{+-}$ *is defined by*

$$F^{+-}(A,B) = F(A,B).$$
$$F^{+-}((f^+,f^-), (g^+,g^-)) = (F(f^-,g^+), F(f^+,g^-)).$$

One problem with the category $\mathbf{C}^{+-}$ is that it is very unlikely to have colimits for all ω-chains. The interesting fact, though, is that the idea upon which definition 10.1.2 is based, works in *every* subcategory $\mathbf{C}^*$ of $\mathbf{C}^{+-}$, provided that *only those functors F such that* $(F(f^-,g^+), F(f^+,g^-))$ *is still a morphism in* $\mathbf{C}^*$ are considered.

Our goal, now, is to find a subcategory $\mathbf{C}^*$ of $\mathbf{C}^{+-}$ such that simple and common properties on **C** (such as, for example, the existence of limits for all diagrams) are enough to guarantee the existence of colimits for all ω-chains in $\mathbf{C}^*$.

In the search for such a category $\mathbf{C}^*$, we can be helped by some intuition.

When in theorem 6.5.2 we considered the ω-diagram $(\{F^i(0)\}_{i\in\omega}, \{F^i(z)\}_{i\in\omega})$, we had in mind that it is a chain of increasingly finer approximations of the limit. Thus, a morphism $F^i(0)$, in a sense, must explain *why* $F^i(z)$ is *less than* $F^{i+1}(z)$: no information must be lost passing from $F^i(z)$ to $F^{i+1}(z)$. Reasonably, we may try to define a subcategory **D** of $\mathbf{C}^{+-}$ whose morphisms express this kind of relation between objects.

Among the subcategories of $\mathbf{C}^{+-}$ that seem to satisfy this condition, an important one is $\mathbf{C}^{\mathbf{Ret}}$, whose morphisms (f^+,f^-) have the property that $f^-\circ f^+ = id$ (in **C**).

$\mathbf{C}^{\mathbf{Ret}}$ is also very attractive because*every* functor F on **C** may be still turned into a covariant functor F^{+-} on $\mathbf{C}^{\mathbf{Ret}}$ by means of definition 10.1.2.

Indeed, consider for example a bifunctor F contravariant in the first component and covariant in the second one. Then one has

$$\begin{aligned} F(f^+,g^-) \circ F(f^-,g^+) &= F((f^+,g^-) \circ (f^-,g^+)) \\ &= F(f^-\circ f^+, g^- \circ g^+) && \text{by composition in } \mathbf{C}^{\mathbf{op}}\mathbf{\times C} \\ &= F(id, id) && \text{as } (f^+,f^-), (g^+,g^-) \text{ are morphisms of } \mathbf{C}^{\mathbf{Ret}} \\ &= id. \end{aligned}$$

Thus $F^{+-}((f^+,f^-), (g^+,g^-)) = (F(f^-,g^+), F(f^+,g^-))$ is well defined as $(F(f^-,g^+), F(f^+,g^-))$ is a retraction pair.

Unfortunately, $\mathbf{C^{Ret}}$ does not seem to suffice for our purposes. For example, up to now, there is no known nontrivial Cartesian closed category $\mathbf{C}$ such that $\mathbf{C^{Ret}}$ has colimits for every ω-chain. Indeed, this is poses a very interesting problem: whether it is possible to find such a category, or prove that it cannot exist.

It is very instructive to understand where the difficulty arises in general.

Suppose that the category **C** has limits for every diagram (this property holds in many interesting CCC's; see chapter 6 for some examples) and let $(\{D_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ be an ω-chain in $\mathbf{C^{Ret}}$. Then $(\{D_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$ is an ω-chain in **C**, and it has a limit $(L,\{\gamma_i\}_{i\in\omega})$. The object L seems to be a good candidate as a limit also for the chain $(\{D_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ in $\mathbf{C^{Ret}}$. Indeed, the following theorem holds:

10.1.3 Theorem *Let* $\{D_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ *be a ω-chain in* $\mathbf{C^{Ret}}$. *If* $(L, \{\gamma_i\}_{i\in\omega})$ *is a limit for* $(\{D_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$ *in* **C**, *then there is a cone* $(L, \{(\delta_i,\gamma_i)\}_{i\in\omega})$ *for* $(\{D_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ *in* $\mathbf{C^{Ret}}$ *(that is, every* γ_i *is a right member of a retraction pair).*

Proof: Fix D_j. For every i define $f_{j,i} : D_j \to D_i$ by:

$$f_{j,i} = f_i^- \circ f_{i+1}^- \circ \ldots \circ f_{j-1}^- \qquad \text{if } i < j$$
$$f_{i,i} = id$$
$$f_{j,i} = f_{i-1}^+ \circ \ldots \circ f_{j+1}^+ \circ f_j^+ \qquad \text{if } i > j .$$

$(D_j, \{f_{j,i}\}_{i\in\omega})$ is a cone for $(\{D_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$, since $f_i^- \circ f_{j,i+1} = f_{j,i}$ as it is easy to check. Thus there exists a unique morphism $\delta_j: D_j \to L$ such that $\forall i\in\omega\ \gamma_i \circ \delta_j = f_{j,i}$. In case $i = j$, $\gamma_j \circ \delta_j = f_{j,j} = id$.

We still have to check that $\forall j\in\omega\ (f_j^+,f_j^-)\circ(\delta_{j+1},\gamma_{j+1}) = (\delta_j,\gamma_j)$.

Now, $f_j^- \circ \gamma_{j+1} = \gamma_j$ by the definition of cone in **C**. In order to prove that $\delta_{j+1} \circ f_j^+ = \delta_j$, note that $\forall i\in\omega\ \gamma_i \circ \delta_{j+1} \circ f_i^+ = f_{j+1,i} \circ f_j^+ = f_{j,j} = \gamma_i \circ \delta_j$, and the result follows by unicity.

By a diagram, for $i > j$:

$$\begin{tikzcd}
D_i & D_{i+1} \arrow[l, "f_i^-"'] & \\
L \arrow[u, "\gamma_i"] \arrow[ur, "\gamma_{i+1}"] & D_j \arrow[l, "\delta_j"] \arrow[ul, "f_{j,i}"] \arrow[u, "f_{j,i+1}"] \arrow[r, "f_j^+"] & D_{j+1} \arrow[ull, "f_{j+1,i}"] \arrow[ul, "f_{j+1,i+1}"'] \arrow[ll, bend left, "\delta_{j+1}"]
\end{tikzcd}$$

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Unfortunately, we have no way to prove that the cone $(L, \{\delta_i,\gamma_i\}_{i\in\omega})$ is universal. Indeed let $(L', \{g_i\}_{i\in\omega})$ be a cone for the ω-chain $(\{D_i\}_{i\in\omega}, \{f_i\}_{i\in\omega})$ in $\mathbf{C}^{\mathbf{Ret}}$. Then $(L',\{g_i^-\}_{i\in\omega})$ is a cone in **C** for $(\{D_i\}_{i\in\omega}, \{f_i^-\}_{i\in\omega})$ and there exists a unique morphism h in **C** from L' to L such that $\forall i\in\omega\ \ h \circ \gamma_i = g_i^-$ and $\gamma_i \circ h = g_i^+$. In other words:

$$\begin{tikzcd}
D_i \arrow[r, shift right, "f_i^+"'] \arrow[d, shift left, "\delta_i"] \arrow[dr, shift left, "g_i^+"] & D_{i+1} \arrow[l, shift right, "f_i^-"'] \arrow[dl, shift left, "\delta_{i+1}"] \arrow[d, shift left, "g_{i+1}^+"] \\
L \arrow[u, shift left, "\gamma_i"] \arrow[ur, shift left, "\gamma_{i+1}"] & L' \arrow[ul, shift left, "g_i^-"] \arrow[u, shift left, "g_{i+1}^-"] \arrow[l, dashed, "h"]
\end{tikzcd}$$

However, there is in general no reasonable way to define a morphism k from L to L' such that $h \circ k = id$. In the next section, we will see a way out of this problem.

10.2 0-Categories

Consider the class of morphisms $\{g_i^+\circ \gamma_i\}_{i\in\omega}$ from L to L' defined by the diagram at the end of the previous section. The morphism $g_i^+ \circ \gamma_i: L\to L'$ describes *how the approximation of L up to the ith-level may be represented inside L'*. Intuitively, one may expect that, in some sense, $g_i^+ \circ \gamma_i \leq g_{i+1}^+ \circ \gamma_{i+1}$. Moreover, if

1. the hom-sets in the category **C** are c.p.o.'s *and*
2. the class $\{g_i^+ \circ \gamma_i\}_{i\in\omega}$ is an ω-chain,

then we could soundly define $k = \cup_{i\in\omega}\{g_i^+ \circ \gamma_i\}$, and k would play the role required at the end of the previous section. This takes us to the notion of "0-category."

10.2.1 Definition *A category **C** is a **0-category** iff*

*i. every hom-set **C**[a,b] is a cpo, with a least element* $0_{a,b}$;

ii. composition of morphisms is a continuous operation with respect to the partial order;

iii. for every $f\in\mathbf{C}[a,b]$, $0_{b,c} \circ f = 0_{a,c}$.

Then, by definition, every 0-category satisfies (1) above. Our next problem is to ensure condition (2). Note first that $g_i^+ \circ \gamma_i = g_{i+1}^+ \circ f_i^+\circ f_i^- \circ \gamma_{i+1}$. Moreover, if $f_i^+\circ f_i^- \leq id$, then $g_i^+ \circ \gamma_i \leq g_{i+1}^+ \circ \gamma_{i+1}$. This suggests the refinement we need in the category $\mathbf{C}^{\mathbf{Ret}}$.

10.2.2 Definition Let **C** be a 0-category, and let $i: D\to E$, $j: E\to D$ be two morphisms in C. Then (i,j) is a **projection pair** (from D to E) iff $j \circ i = id_D$ and $i \circ j \leq id_E$. (If (i,j) is a projection pair, i is an **embedding** and j is a **projection**.)

Exercises Prove the following statements:

1. If (i,j) is a projection pair from D to E, and (i',j') is another projection pair for the same two objects, then $i \leq i'$ iff $j \geq j'$.
2. Every embedding i has a unique associated projection $j = i^R$; conversely every projection j has a unique associated embedding $i = j^L$.

10.2.3 Definition *Let **C** be a 0-category. The category $\mathbf{C^{Prj}}$ has the same objects as **C**, and projection pairs as morphisms.*

Remark $\mathbf{C^{Prj}}$ is a subcategory of $\mathbf{C^{Ret}}$, and thus also of $\mathbf{C^{+-}}$. By the fact that every embedding i has a unique associated projection $j = i^R$ (and, conversely, every projection j has a unique associated embedding $i = j^L$), $\mathbf{C^{Prj}}$ is isomorphic to a subcategory $\mathbf{C^E}$ of **C** that has embeddings as morphisms (as well to a subcategory $\mathbf{C^P}$ of **C** which has projections as morphisms). We prefer to work in $\mathbf{C^{Prj}}$ since it looks more natural and carries more explicit information. Note, however, the following dualities: $(\mathbf{C^E})^{op} \cong \mathbf{C^P} \cong (\mathbf{C}^{op})^{\mathbf{E}}$ (and, of course, $(\mathbf{C^P})^{op} \cong \mathbf{C^E} \cong (\mathbf{C}^{op})^{\mathbf{P}}$).

Exercise Let **C** be a 0-category with terminal object t. Prove that t is terminal in $\mathbf{C^{Prj}}$ too. (Use property 10.2.1.(iii) in the definition of 0-category).

10.2.4 Theorem *Let **C** be a 0-category with all ωop-limits. Let $(\{D_i\}_{i\in\omega},\{(f_i^+,f_i^-)\}_{i\in\omega})$ be an ω-chain in $\mathbf{C^{Prj}}$ (and thus in $\mathbf{C^{Ret}}$) and $(L, \{(\delta_i,\gamma_i)\}_{i\in\omega})$ be the cone in $\mathbf{C^{Ret}}$ defined by theorem 10.1.3. Then $(L, \{(\delta_i,\gamma_i)\}_{i\in\omega})$ is a cone also in $\mathbf{C^{Prj}}$. Moreover it is universal in this category.*

Proof In order to prove that $(L, \{(\delta_i,\gamma_i)\}_{i\in\omega})$ is a cone in $\mathbf{C^{Prj}}$ we must show that, $\forall i\in\omega$, $\delta_i \circ \gamma_i \leq id$. Note that $\forall i\in\omega$ $\delta_i \circ \gamma_i = \delta_{i+1} \circ f_i^+ \circ f_i^- \circ \gamma_{i+1} \leq \delta_{i+1} \circ \gamma_{i+1}$. Thus, $\{\delta_i \circ \gamma_i\}_{i\in\omega}$ is a chain and its limit $\Theta = \cup_{i\in\omega}\{\delta_i \circ \gamma_i\}$ must exist. We prove that $\Theta = id$ and, thus, that $\forall i\in\omega$ $\delta_i \circ \gamma_i \leq \Theta = id$. $\Theta = id$ since Θ is a mediating morphism between $(L, \{\gamma_i\}_{i\in\omega})$ and itself in **C**. Indeed, $\forall j\in\omega$,

$$\begin{aligned}\gamma_j \circ \Theta &= \gamma_j \circ \cup_{i\in\omega}\{\delta_i \circ \gamma_i\}\\ &= \gamma_j \circ \cup_{i \geq j}\{\delta_i \circ \gamma_i\}\\ &= \cup_{i \geq j}\{(\gamma_j \circ \delta_i) \circ \gamma_i\}\\ &= \cup_{i \geq j}\{f_{i,j} \circ \gamma_i\}\end{aligned}$$

$$= \cup_{i \geq j}\{f_{i,j} \circ \gamma_i\}$$
$$= \gamma_j.$$

We prove next that the cone $(L, \{(\delta_i,\gamma_i)\}_{i\in\omega})$ is universal in $\mathbf{C}^{\mathbf{Prj}}$.

Let $(L',\{(g_i^+,g_i^-)\}_{i\in\omega})$ be another cone for $(\{D_i\}_{i\in\omega}, \{(f_i^+,f_i^-)\}_{i\in\omega})$. That is,

$$\forall i\in\omega \quad g_i^+\circ \gamma_i = g_{i+1}^+\circ f_i^+\circ f_i^- \circ \gamma_{i+1} \leq g_{i+1}^+\circ \gamma_{i+1}$$
$$\forall i\in\omega \quad \delta_i \circ g_i^- = \delta_{i+1} \circ f_i^+\circ f_i^- \circ g_{i+1}^- \leq \delta_{i+1} \circ g_{i+1}^-.$$

Define then

$$h = \cup_{i\in\omega}\{g_i^+ \circ \gamma_i\} : L \to L'$$
$$k = \cup_{i\in\omega}\{\delta_i \circ g_i^-\} : L' \to L.$$

Observe that (h,k) is a projection pair, for

$$\begin{aligned} k \circ h &= \cup_{i\in\omega}\{\delta_i \circ g_i^-\} \circ \cup_{i\in\omega}\{g_i^+ \circ \gamma_i\} \\ &= \cup_{i\in\omega}\{\delta_i \circ (g_i^- \circ g_i^+) \circ \gamma_i\} \\ &= \cup_{i\in\omega}\{\delta_i \circ \gamma_i\} \\ &= \Theta = id \end{aligned}$$

and

$$\begin{aligned} h \circ k &= \cup_{i\in\omega}\{g_i^+ \circ \gamma_i\} \circ \cup_{i\in\omega}\{\delta_i \circ g_i^-\} \\ &= \cup_{i\in\omega}\{g_i^+ \circ (\gamma_i \circ \delta_i) \circ g_i^-\} \\ &= \cup_{i\in\omega}\{g_i^+ \circ g_i^-\} \\ &\leq id. \end{aligned}$$

Moreover, (h,k) is a mediating morphism between $(L, \{(\delta_i,\gamma_i)\}_{i\in\omega})$ and $(L',\{(g_i^+,g_i^-)\}_{i\in\omega})$, since $\forall j\in\omega$

$$\begin{aligned} (h,k) \circ (\delta_j,\gamma_j) &= (h \circ \delta_j , \gamma_j \circ k) \\ &= (\cup_{i\in\omega}\{g_i^+ \circ \gamma_i\} \circ \delta_j , \gamma_j \circ \cup_{i\in\omega}\{\delta_i \circ g_i^-\}) \\ &= (\cup_{i \geq j}\{g_i^+ \circ \gamma_i \circ \delta_j\} , \cup_{i \geq j} \{ \gamma_j \circ \delta_i \circ g_i^-\}) \\ &= (\cup_{i \geq j}\{g_i^+ \circ f_{j,i}\} , \cup_{i \geq j} \{ f_{i,j} \circ g_i^-\}) \\ &= (g_j^+,g_j^-) \end{aligned}$$

(h,k) is unique, because, if (h',k') is another mediating morhism, then

$$\begin{aligned} (h',k') &= (h'\circ id, id \circ k') \\ &= (h'\circ \Theta, \Theta \circ k') \\ &= (h'\circ \cup_{i\in\omega}\{\delta_i \circ \gamma_i\}, \cup_{i\in\omega}\{\delta_i \circ \gamma_i\} \circ k') \\ &= (\cup_{i\in\omega}\{h'\circ \delta_i \circ \gamma_i\}, \cup_{i\in\omega}\{\delta_i \circ \gamma_i \circ k'\}) \\ &= (\cup_{i\in\omega}\{g_i^+ \circ \gamma_i\}, \cup_{i\in\omega}\{\delta_i \circ g_i^-\}) \\ &= (h,k) . \blacklozenge \end{aligned}$$

A useful characterization of ω-colimits in the category $\mathbf{C}^{\mathbf{Prj}}$ is the following:

10.2.5 Proposition *The cone* $(L, \{(\delta_i,\gamma_i)\}_{i\in\omega})$ *for the ω-chain* $(\{D_i\}_{i\in\omega}, \{(f_i^+,f_i^-)\}_{i\in\omega})$ *in* $\mathbf{C^{Prj}}$ *is universal iff* $\Theta = \cup_{i\in\omega}\{\delta_i \circ \gamma_i\} = id$.

Proof Exercise. ♦

Up to now, we have shown that, if **C** is a 0-category with all ωop-limits, then the category $\mathbf{C^{Prj}}$ has colimits for every ω-chain.

The next point is to understand what we have lost with regard to the possibility of applying the construction in definition 10.1.2, which turns contravariant functors into covariant ones. Indeed, there is no reason to believe that the functor F^{+-} of this definition transforms projection pairs within projection pairs.

Recall now that a two-argument endofunctor F over **C**, which is contravariant in the first argument and covariant in the second one, has type $F: \mathbf{C^{op}}\times\mathbf{C} \to \mathbf{C}$.

10.2.6 Definition *Let **C** be a 0-category. A functor* $F: \mathbf{C^{op}}\times\mathbf{C} \to \mathbf{C}$ *is **locally monotonic** iff it is monotonic on the hom-sets; that is, for* $f, f'\in\mathbf{C^{op}}[A, B]$ *and* $g, g'\in\mathbf{C}[C, D]$, *one has*

$$f \le f' ,\ \ g \le g' \ \ \Rightarrow\ \ F(f,g) \le F(f',g').$$

10.2.7 Proposition *If* $F: \mathbf{C^{op}}\times\mathbf{C} \to \mathbf{C}$ *is locally monotonic and* (f^+,f^-), (g^+,g^-) *are projection pairs, then also* $F^{+-}((f^+,f^-),(g^+,g^-))$ *is also a projection pair.*

Proof: By definition $F^{+-}((f^+,f^-),(g^+,g^-)) = (F(f^-,g^+), F(f^+,g^-))$. Compute then

$$\begin{aligned} F(f^+,g^-) \circ F(f^-,g^+) &= F((f^+,g^-) \circ (f^-,g^+)) \\ &= F(f^- \circ f^+, g^- \circ g^+) \\ &= F(id, id) \\ &= id \end{aligned}$$

and

$$\begin{aligned} F(f^-,g^+) \circ F(f^+,g^-) &= F((f^-,g^+) \circ (f^+,g^-)) \\ &= F(f^+ \circ f^-, g^+ \circ g^-) \\ &\le F(id, id) \\ &= id. \end{aligned}$$ ♦

The last step is to see if we can find some simple condition on the functor F in **C** such that the associated functor F^{+-} in $\mathbf{C^{Prj}}$ is ω-continuous.

10.2.8 Definition *Let **C** be a 0-category. A functor* $F: \mathbf{C^{op}}\times\mathbf{C} \to \mathbf{C}$ *is **locally continuous (0-functor)** iff it is ω-continuous on the hom-sets. That is, for every directed set* $\{f_i\}_{i\in\omega}$ *in* $\mathbf{C^{op}}[A, B]$, *and every directed set* $\{g_i\}_{i\in\omega}$ *in* $\mathbf{C}[C, D]$, *one has*

$$F(\cup_{i\in\omega}\{f_i\}, \cup_{i\in\omega}\{g_i\}) = \cup_{i\in\omega} F(f_i,g_i) .$$

Of course, if F is locally continuous, then it is also locally monotonic.

Exercise Prove that the composition of two locally nonotonic (continuous) functors is still monotonic (continuous).

10.2.9 Theorem *Let* **C** *be a 0-category with all ωop-limits. Let also* $F: \mathbf{C}^{op}\times\mathbf{C}\to\mathbf{C}$ *be a locally continuous functor. Then the functor* $F^{+-}: \mathbf{C}^{\mathbf{Prj}}\times\mathbf{C}^{\mathbf{Prj}}\to\mathbf{C}^{\mathbf{Prj}}$ *is ω-continuous.*

Proof Let $(\{A_i\}_{i\in\omega}, \{(f_i^+,f_i^-)\}_{i\in\omega})$ and $(\{B_i\}_{i\in\omega}, \{(g_i^+,g_i^-)\}_{i\in\omega})$ be two ω-chains in $\mathbf{C}^{\mathbf{Prj}}$ and let $(L, \{(\rho_i^+,\rho_i^-)\}_{i\in\omega})$ and $(M, \{(\sigma_i^+,\sigma_i^-)\}_{i\in\omega})$ be the respective limits. Then, by proposition 10.2.5,

$$\Theta = \cup_{i\in\omega}\{\rho_i^+ \circ \rho_i^-\} = \mathrm{id};$$

$$\Psi = \cup_{i\in\omega}\{\sigma_i^+ \circ \sigma_i^-\} = \mathrm{id}.$$

We must show that

$$(F^{+-}(L,M), \{F^{+-}((\rho_i^+,\rho_i^-), (\sigma_i^+,\sigma_i^-))\}_{i\in\omega}) = (F(L,M), \{F(\rho_i^-,\sigma_i^+), F(\rho_i^+,\sigma_i^-))\}_{i\in\omega})$$

is a limit in $\mathbf{C}^{\mathbf{Prj}}$ for the ω-chain

$$(\{F^{+-}(A_i,B_i)\}_{i\in\omega}, \{F^{+-}((f_i^+,f_i^-), (g_i^+,g_i^-))\}_{i\in\omega}).$$

It is clearly a cone, by the property of functors. We show that it is universal by proving that $\cup_{i\in\omega}\{ F(\rho_i^-,\sigma_i^+) \circ F(\rho_i^+,\sigma_i^-) \} = \mathrm{id}$. The result then follows by proposition 10.2.5. We have the following:

$$\begin{aligned}\cup_{i\in\omega}\{ F(\rho_i^-,\sigma_i^+) \circ F(\rho_i^+,\sigma_i^-)\} &= \cup_{i\in\omega}\{ F(\rho_i^+ \circ \rho_i^-, \sigma_i^+ \circ \sigma_i^-) \}\\ &= F(\cup_{i\in\omega}\{\rho_i^+ \circ \rho_i^-\}, \cup_{i\in\omega}\{ \sigma_i^+ \circ \sigma_i^- \})\\ &= F(\Theta, \Psi)\\ &= F(\mathrm{id}, \mathrm{id})\\ &= \mathrm{id}. \ \blacklozenge\end{aligned}$$

In conclusion, we have described a way to turn arbitrary functors into covariant functors on suitably derived categories. Then we set the condition under which it is possible to obtain ω-continuous functors. The solution of equations, such as (*) at the beginning of this chapter, is thus immediately found for these functors.

Example In the introduction to this section we mentioned the important categorical equation $X \cong A+(X\to X)$. If we wish to find a solution to equations of this kind in some category **C** based on c.p.o.s (such as **CPO**, **CPOS**, **Scott Domains**, and so on), we are generally forced to relax the interpretation of at least one of the two symbols + and →. Indeed, for their nature, all these categories usually have fixpoints for all objects, and we know that this is inconsistent with having at the same time coproducts and cartesian closedness. A typical way for avoiding the problem is to content

ourselves with the interpretation of + as a weak coproduct. A **weak** coproduct is essentially defined as a coproduct in definition 2.2.6, but no unicity is requested for the commuting arrow h.
The category **CPO** of c.p.o.'s with least (bottom) element and continuous functions for morphisms is a 0-category with respect to the pointwise ordering of morphisms. **CPO** is a CCC with weak coproducts A+B given by the coalesced sum (i.e. by identifying the two bottom elements of A and B). Since it has coequalizers for every pair of objects, it has limits for every diagram.(see chapters 2 and 6). The functors $A+\_ : \mathbf{CPO} \to \mathbf{CPO}$ and $\to\ : \mathbf{CPO}\times\mathbf{CPO} \to \mathbf{CPO}$, respectively defined by:

$$A+\_\,(B) = A+B,\quad A+\_\,(f) = id_A + f,$$
$$\to(A,B) = B^A,\quad \to(f, g) = \lambda h.\ g \circ h \circ f,$$

are both locally continuous.

The diagonal functor $\Delta : \mathbf{CPO} \to \mathbf{CPO}\times\mathbf{CPO}$, defined by $\Delta(A) = (A,A)$ and $\Delta(f) = (f,f)$, is locally continuous too. Thus we can apply theorem 10.2.9 and conclude that the associated functors

a. $(A+\_)^{+-} : (\mathbf{CPO})^{\mathbf{Prj}} \to (\mathbf{CPO})^{\mathbf{Prj}}$
   $(A+\_)^{+-}(f^+, f^-) = (id_A+f^+, id_A+f^-)$

b. $(\to)^{+-} : (\mathbf{CPO})^{\mathbf{Prj}}\times(\mathbf{CPO})^{\mathbf{Prj}} \to (\mathbf{CPO})^{\mathbf{Prj}}$,
   $(\to)^{+-}((f^+, f^-), (g^+,g^-)) = (\lambda h.\ g^+ \circ h \circ f^-, \lambda h.\ g^- \circ h \circ f^+)$

c. $(\Delta)^{+-} : (\mathbf{CPO})^{\mathbf{Prj}} \to (\mathbf{CPO})^{\mathbf{Prj}}\times(\mathbf{CPO})^{\mathbf{Prj}}$
   $(\Delta)^{+-}(f^+, f^-) = ((f^+, f^-), (f^+, f^-))$

are ω-continuous. But composition of ω-continuous functors is still a ω-continuous functor; thus, the functor $F = (A+\_)^{+-} \circ (\to)^{+-} \circ (\Delta)^{+-} : (\mathbf{CPO})^{\mathbf{Prj}} \to (\mathbf{CPO})^{\mathbf{Prj}}$ is ω-continuous. Explicitly, F is defined by

$$F(X) = A+X^X$$
$$F(f^+,f^-) = (id_A+ \lambda h.\ f^+ \circ h \circ f^-, id_A+ \lambda h.\ f^- \circ h \circ f^+).$$

Thus, for every A, there exists X such that $X \cong A+X^X$.

References For an early computer scientific introduction to recursive domain equations, the reader should consult Stoy (1977). The first solution to the problem of finding a domain isomorphic with its own function space, as required for the type-free λ-calculus, was given in Scott (1972) which basically started the mathematical discussion on recursive definitions of data types and, more generally, the so called area of "denotational semantics". The categorical approach exposed in the present chapter is a direct generalization of Scott's method and is essentailly due to Wand (1979), Lehmann and Smith (1981) and Smith and Plotkin (1982). An introductory presentation may be also found in Plotkin (1978). Gunter (1985) investigates the notion of embedding as a particular case of adjunction and, thus, sets the base for an interesting generalization of the categorial approach.