CHAPTER ONE: SETS

1 Introduction

Scientific theories usually do not directly describe the natural phenomena under investigation, but rather a mathematical idealization of them that abstracts away from various complicating factors. For example, a theory about how the earth, the sun, and the moon move under mutual gravitation might ignore such complications as the sizes of the three bodies, friction arising from the presence of interstellar dust, the gravitational force exerted by other planets and stars, or relativistic effects that become significant only as the velocities of the bodies in question approach the speed of light. In the mathematical idealization, the time might be represented by a real number; the mass of each of the three bodies by a positive real number; its location in space (or more precisely the location of its center of gravity) at a particular time by three real numbers (the x, y, and z coordinates relative to a coordinate system); its velocity at a particular time by three more real numbers; the state of the three-body system at a given time (real number) t by the 18 real numbers that specify the locations and velocities of the three bodies at time t; and the evolution of the system over time by 18 functions that give the value of each of these 18 parameters at each time t. And the theory itself is a mathematical specification of which evolutions ('paths') through 18-dimensional Euclidean space) are possible. Armed with such a theory, we can predict, given the state of the system at a given time t_0, what state it will be in at any future time t_1.

Linguistic theories make predictions not about celestial bodies, but rather about natural languages, for example: how their words can sound; how their words can be combined into phrases; what meanings they can express; which natural-language arguments are judged valid; or how the meanings of sentences can be related to the meanings of the words they contain. As it turns out, the kinds of mathematical entities that have proven to be useful for representing such things (words, phrases, sentences, their meanings, valid arguments, etc.) are not real numbers or real-valued functions, but rather discrete (roughly, non-continuous) things such as natural numbers, strings, trees, algebras, formal languages, and proof systems. In linguistics these mathematical idealizations are often called **representations** or **models** of the phenomena in question. In this book the first of these terms will be preferred, to avoid confusion with a different, technical, use of the term "model" (in the sense of an interpretation of a logical theory) to be introduced in later chapters. For example, phrases (roughly speaking, multi-word expressions, including sentences) are often represented as (mathematical) *trees*; phonemes (roughly, minimal units of linguistic sound) as (mathematical) *graphs* of a certain kind (*feature structures*); the sequences of sounds that make up (the phonology of) words as (mathematical) *strings* of (representations of) phonemes; and linguistic meanings as (mathematical) *functions* of various kinds. (Note that it is typical for technical mathematical terms, such as *tree*, *string*, and *function*, to have other, nonmathematical meanings!)

In order to have a clear understanding of what these different kinds of mathematical entities are and why they are able to serve as linguistic representations, we will start out with an overview of **set theory**. Sets are basic mathematical entities whose existence is taken for granted by most mathematicians, and set theory begins with certain assumptions about them. Set theory is the workspace that most mathematicians work in; but more importantly for us, it is where the idealized representation of natural phenomena by linguists and other scientists is carried out. That is, sets are used to construct the represesentations of natural-language phenomena that linguistic theories talk about. In fact, all the kinds of linguistic representations mentioned above (trees, graphs, strings, and functions) are themselves sets.

2 Sets and Membership

We assume that there exist things which we call **sets**, and that there is a relationship, called **membership**, which either does or does not hold of any two sets. That is, if A is a set and B is a set, then either A **is a member of** B (written $A \in B$) or A is not a member of B (written $A \notin B$). There are many ways to say this. The members of a set are also called its **elements**, and instead of saying A is a member of B, we often say it **belongs to** B, or is **in** B, or is **contained in** B. Intuitively, sets can be thought of as something like collections, where the members are the things collected, or as invisible baskets, with the members being the things in the baskets. But set theory will never tell us what sets are; they are basic and cannot be reduced to, or explained in terms of, more basic things that are not sets. That is, they are the **unanalyzed primitives** of set theory.

We will make certain assumptions about how membership works based on these intuitions, and then try to ascertain what follows from them. These assumptions themselves, together with the facts that follow from them, constitute **set theory**. To be slightly more precise, they are *a* set theory, since some assumptions about how sets should work are controversial. In this chapter, we will make some of the most generally accepted of these assumptions explicit and consider some of their consequences. (In due course we will also consider some of the more controversial assumptions about how sets work.)

For the time being, we will state our assumptions about sets in English, and conduct our reasoning about what follows from these assumptions using intuitively valid English arguments called **informal proofs**. Later on we will see that it is possible to **formalize** the assumptions of set theory with the help of specialized symbolic systems (formal logics, such as predicate logic). In that case the formalized counterparts of the assumptions are called **axioms**; the additional formulas that follow from them are called **theorems**; and the formalized counterparts of the English arguments we make to justify these theorems are called **formal proofs**.

In fact, informal (but precise) natural-language reasoning is the norm amomg mathematicians and natural scientists. Usually they don't bother to formalize proofs unless they are studying proofs as mathematical objects in their own right. Later we will have occasion to do just that, for the (perhaps surprising) reason that linguistic expressions and their meanings can themselves be thought of as proofs in certain kinds of logical systems.

In ascertaining what follows from the assumptions we will make about sets and membership, the reasoning we use will be pretty much the same kind of reasoning we use when we draw conclusions from assumptions about ordinary things, e.g. kitchen appliances, furniture, people, etc. (There are, however, some ways of arguing and ways of expressing arguments that are typical of mathematical discourse, which we will look at more closely in the following chapter.) In practice, mathematics consists of more or less ordinary reasoning about not-so-ordinary things. The upshot, seemingly paradoxical, is that so-called formal linguistics is mostly done within *informal* set theory. The resolution of the apparent paradox is that even informal set theory is more precise and explicit than linguistics that uses no set theory at all.

Now we're ready to start introducing our basic assumptions about sets, and considering some of their consequences.

3 Basic Assumptions about Sets

We are already assuming that there *are* sets, and that if A and B are sets, then either $A \in B$ or $A \notin B$. But to be able to *do* anything with sets, we need to make some assumptions about how they work. The assumptions we make in this chapter are the ones that are generally considered the most basic, intuitively plausible, and uncontroversial. Later we will add a few more (but not many more), including some that not all mathematicians are entirely comfortable with. We give each assumption a name, to make it easy to refer to.

Assumption 1 (Extensionality): If A and B have the same members, then they are the same set (written $A = B$).

Note that in stating this assumption, we did not bother to mention that A and B are sets. That is because we've already established that we are now doing (informal) set theory, and in set theory, the only things being talked about are sets. Note also that we do not have to explicitly assume (though it is true) that if A and B do *not* have the same members, then they are *not* the same set (written $A \neq B$). That's because, if they were the same set, then everything about them, including what members they have, would be the same. This reasoning is no different than the kind of reasoning we would use to conclude (given that A and B are people), that if A and B do not have the same blood type, then they cannot be the same person: if they were the same person, everything about them—including their blood types—would be the same.

If every member of A is a member of B, we say that A is a **subset** of B, or, alternatively, that A is **included in** B), written $A \subseteq B$. Note that if $A \subseteq B$, B might have members that are not in A. On the other hand, if both $A \subseteq B$ and $B \subseteq A$, then it follows from Extensionality that $A = B$. If $A \subseteq B$ but $A \neq B$ then we say A is a **proper** subset of B, written $A \subsetneq B$.

Assumption 2 (Empty set): There is a set with no members.

Note that from this assumption together with Extensionality we can conclude that the there is *only* one set with no members. We call this set the **empty** set. The empty set is usually denoted by the symbol '$\emptyset$'. But later, we'll sometimes write it as '0' (the symbol for the number zero), because according to the most usual way of doing arithmetic within set theory (which we'll get to in Chapter 4), the number zero and the empty set are the same thing (in spite of what you may have been taught in other math classes!).

Now so far, we have no basis for concluding that there are any sets other than the empty set, not even sets with only one member. For example, we are not even able to make a valid argument that there is a set with $\emptyset$ as its only member. We remedy this situation by adding a few more assumptions, beginning with the following:

Assumption 3 (Pairing): For any sets A and B, there is a set whose only members are A and B.

Note that, because of Extensionality again, there is *only* one set whose only members are A and B, which we write as $\{A, B\}$, Of course we could just as well have called this set $\{B, A\}$. More generally, we will notate any nonempty finite set by listing its members, separated by commas, between curly brackets, in any order. (In Chapter 5, we'll get clear about what we mean when we say a set is 'finite', but for now we'll just rely on intuition). Notice that nothing rules out the possibility that A and B are the same set, so it follows from pairing that for any set A there is a set whose only member is A, namely $\{A, A\}$. Of course, once we realize this, then we might as well just call it $\{A\}$ rather than $\{A, A\}$: repetitions inside the curly brackets don't make any difference because for any given set, either A is a member of it or it isn't; it doesn't make any sense to talk about *how many times* one set is a member of another.

A set with only one member is called a **singleton**. A special case of singleton sets is the set $\{0\}$ whose only member is 0. This set is also called 1, because according to the usual way of doing arithmetic within set theory, it is the same as the number one. Going one step further, we can use Pairing again to form the set $\{0, 1\}$, also known as 2. There is a general pattern here, which we will explain in Chapter 4.

Assumption 4 (Union): For any set A, there is a set whose members are those sets which are members of (at least) one of the members of A.

Once again, Extensionality ensures the uniqueness of such a set, which is called the **union** of A, written $\bigcup A$. As a special case, if $A = \{B, C\}$, then $\bigcup A$ is the set each of whose members is in either B or C (or both). This set is usually written $B \cup C$. Note that in general this is not the same thing as $\{B, C\}$!

For any set A, the **successor** of A, written $s(A)$, is the set $A \cup \{A\}$. That is, $s(A)$ is the set with the same members as A, except that A itself is also a member of $s(A)$.[1] For example, 1 is the successor of 0, and 2 is the successor of 1.

Assumption 5 (Powerset): For any set A, there is a set whose members are the subsets of A.

[1]Nothing we have said rules out the possibility that $A \in A$, in which case $A = s(A)$. However, the most widely used set theory (called **Zermelo-Fraenkel** set theory) includes an assumption (called **Foundation**) which does rule out this possibility. We will not assume Foundation in this book.

Yet again, Extensionality guarantees the uniqueness of such a set. We call it the **powerset** of A, written $\wp(A)$. It's important to realize that $\wp(A)$ is usually not the same set as A. That's because usually the subsets of a set are not the same as the members of the set. For example, 0 is a subset of 0 (in fact, every set is a subset of itself), but obviously 0 is not a member of 0 (since 0 is the empty set).

4 Russell's Paradox and Separation

Why do we need the powerset assumption? Why don't we just *define* $\wp(A)$ to be the set of all subsets of A? The answer is that the other assumptions we have made so far do not seem to enable us to conclude that there actually *is* such a set. More generally, whenever one says "the set of all sets such that blah-blah-blah", there is no guarantee that the assumptions one has made about sets enable one to conclude that there actually is a set meeting that description. That may seem counterintuitive, but, perhaps surprisingly, there is a knockdown argument that there is no such guarantee, which was discovered by the philosopher and mathematician Bertrand Russell.[2]

The argument runs as follows. Consider the description "the set of all sets which are not members of themselves." Suppose for a moment there were such a set, called R. Then would R be member of R? Well, either it is or it isn't. In the first case, we see right away that R cannot be a member of R. And in the second case, we see right away that R must be a member of R. Either way, we arrive at a contradiction, and so our temporary assumption that there is a set whose members are the sets which are not members of themselves must have been false. This argument is called **Russell's Paradox**.

Russell's Paradox shows that, in general, we cannot assume that, for any set description, we can take for granted the existence of a set meeting that description. However, there is a more cautious assumption that proves to be extremely useful and which so far has not been shown to result in paradox.

Assumption 6 (Separation): If A is a set and $P[x]$ is a condition on x (where x is a variable that ranges over sets), then there is a set, written $\{x \in A \mid P[x]\}$, whose members are all the x in A that satisfy $P[x]$.

Separation is so-called because, intuitively, we are separating out from A some members that are special in some way, and collecting them together

[2]Russell made this argument in a famous letter written in 1902 to Gottlob Frege, another philosopher and mathematician, whose accomplishments include the invention of predicate calculus and of modern linguistic semantics.

into a set. We call Separation an assumption, but to be more precise it is an assumption **schema**: for each condition $P[x]$, we get a different separation assumption. For the moment we remain deliberately vague about what we mean by "a condition on x". (We'll clear this up in due course when we formalize set theory using predicate logic.) For the moment, the easiest way to get an idea of what we mean by a condition on x is to look at some examples.

First, suppose we have two sets A and B. Then by taking $P[x]$ to be the condition $x \in B$, Separation guarantees the existence of the set consisting of those members of A which are also in B. This set is called the **intersection** of A and B, written $A \cap B$. A and B are said to **intersect** if $A \cap B$ is non-empty; otherwise they are said to be **disjoint**. A set is called **pairwise disjoint** if no two distinct members of it intersect.

Second, by taking $P[x]$ to be the condition $x \notin B$, Separation guarantees the existence of the set consisting of those members of A which are *not* in B. This set, called the **complement of** B **relative to** A, or the **set difference of** A **and** B, is written $A \setminus B$.

A rather different application of Separation shows that there can be no set of all sets. For suppose there were; then applying Separation to it using the condition $x \notin x$, we would have the set of all sets which are not members of themselves. But as we already saw (Russell's Paradox), there can be no such set.

5 Ordered Pairs and Cartesian (Co-)Products

Sets do not embody any notion of order: $\{A, B\} = \{B, A\}$. But for linguistic applications, clearly we cannot escape from dealing with order! For example, we cannot describe the phonology of a word without specifying the order of the phonemes in it, not can we fully describe a sentence without specifying the order of its words. One way we might imagine responding to this need is simply to *assume* that for any A, B, there is an *ordered pair* $\langle A, B \rangle$. But what properties should we assume that ordered pairs have? Perhaps surprisingly, it turns out that once we have gotten clear about how ordered pairs should work, the assumptions we have already made about sets enable us to conclude that sets with the desired properties already exist. So we do not need to make any further assumptions in order to have ordered pairs.

In fact, the crucial property of ordered pairs, from which their usefulness derives, that they are *uniquely determined by their components*, in the sense that $\langle A, B\rangle = \langle C, D\rangle$ if and only if $A = C$ and $B = D$. Any way of defining the notion of ordered pair that results in their demonstrably having this property will suffice. The approach we will adopt here is the standard one, which is to define the **ordered pair** of A and B, written $\langle A, B\rangle$, to be the set $\{\{A\}, \{A, B\}\}$. A and B are called, respectively, the **first** and **second component** of $\langle A, B\rangle$. Notice that an ordered pair has either one or two members. In the first case, which arises when $A = B$, the ordered pair is just $\{\{A\}\}$, and both components are A. In the second case, the ordered pair has two members, one with one member and one with two members. In that case, the first component of the pair is the one that belongs to the set with one member, and the second component is the member of the two-member set which is not the member of the one-member set.

Given two sets A and B, it is also useful to have the notion of the **cartesian product** of A and B, written $A \times B$, which is supposed to be the set of all ordered pairs $\langle C, D\rangle$ such that $C \in A$ and $D \in B$. As it turns out, we do not have to *assume* that cartesian products exist, because their existence follows from Separation. (Showing this is left as an exercise.) A and B are called the **factors** of $A \times B$.

Having defined ordered pairs, we can now proceed to define an **ordered triple** to be an ordered pair whose first component is an ordered pair:

$$\langle A, B, C\rangle =_{\text{def}} \langle\langle A, B\rangle, C\rangle$$

and correspondingly the threefold cartesian product:

$$A \times B \times C =_{\text{def}} (A \times B) \times C$$

The definitions can be extended to quadruples, quintuples, etc. in the obvious way. Special cases of cartesian products, called cartesian **powers**, are ones where the factors are all the same set A. These are notated with parenthesized "exponents" (superscripts), e.g. $A^{(2)} = A \times A$, $A^{(3)} = A \times A \times A$, etc. Additionally, we define $A^{(1)}$ to be A, and we define $A^{(0)}$ to be 1. This last definition is less mysterious than it appears to be, but we will be in a better position to explain the motivation for it a little later. (It is actually closely related to the reason that $n^0 = 1$ in arithmetic, but for some readers, that may seem equally mysterious.)

Less well known than cartesian product, but also important in some of our applications, is the notion of the **cartesian coproduct**, also called the **disjoint union**) of A and B, written $A + B$. This is supposed to be the set of ordered pairs $\langle C, D\rangle$ such that either $C = 0$ and $D \in A$, or $C = 1$ and $D \in B$. As with cartesian products, the existence of cartesian coproducts can be demonstrated using Separation A and B are called the **cofactors** of $A + B$.

Intuitively, $A + B$ is the union of two sets, “copies” of A and B respectively, and these copies are disjoint, even if A and B are not. As with cartesian products, there is a straightforward extension to more than two cofactors. For the case of identical cofactors (called cartesian **copowers**), there does not seem to be a standard notation; here we write $A_{(n)}$, which, intuitively, is the union of n pairwise disjoint copies of A. So it should not come as much of a surprise that $A_{(1)}$ is defined to be A and $A_{(0)}$ is defined to be 0.