

CHAPTER FIVE: INFINITIES

Two sets A and B are said to be **equinumerous**, written $A \approx B$, iff there is a bijection from A to B . It follows that a set is finite iff it is equinumerous with a natural number.

As we've observed (Chapter 3, problem 1), equinumerosity is an equivalence relation on the powerset of any set. It is not hard to show that for any set A , $\wp(A) \approx 2^A$: the bijection in question maps each subset of A to its characteristic function (with respect to A).

Intuitively speaking, equinumerosity may seem to amount to "having the same number of members". As we soon will see, this intuition is essentially on the mark in the case of finite sets. But when the sets involved are infinite, intuition may fail us. For example, all the following sets can be shown to be (pairwise) equinumerous: ω , $\omega \times \omega$, the set \mathbb{Z} of integers, and the set \mathbb{Q} of rational numbers.¹

Not all infinite sets are equinumerous! To put it imprecisely but suggestively, there are "different sizes of infinity". For example, as Cantor famously proved, $\omega \not\approx I$, where I is the set of real numbers from 0 to 1. It is beyond the scope of this book to consider how the real numbers are modelled set-theoretically, but for our purposes it will suffice to think of I as the set of "decimal expansions", i.e. the set of functions from ω to 10 ($= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) excluding the ones which for some natural number n assign 9 to every natural number greater than or equal to n .² The proof is surprisingly simple: suppose f is an injection from ω to I . Then f cannot be a surjection. To see why, let r be the member of I (i.e. the function from ω to 10) which, for each $n \in \omega$, maps n to 6 if $f(n) = 5$ and maps n to 5 otherwise. A moment's thought shows that r cannot be in the range of f !

Theorem: For any set A , $A \not\approx \wp(A)$.

Proof: Let g be a function from A to $\wp(A)$. We will show g cannot be surjective. To this end, let $B = \{x \in A \mid x \notin g(x)\}$. Then obviously $B \in \wp(A)$. But B cannot be in the range of g . For suppose it were. In that case there would exist a $y \in A$ such that $B = g(y)$. But then $y \in B$ iff $y \notin g(y)$. In other words, $y \in B$ iff $y \notin B$, a contradiction. \square

A set is said to be **Dedekind infinite** iff it is equinumerous with a proper

¹Actually proving all these things would of course require us to model 'the integers' and 'the rationals' as sets. There are standard ways of doing that, but limitations of space and time prevent us from spelling them out here.

²We can omit these because they have alternative decimal expansions, e.g. .7999... represents the same real number as .8000...

subset of itself. (Contrast this with the definition that a set is **infinite** if it is not equinumerous with any natural number.)

Theorem: No natural number is Dedekind infinite.

Proof: Exercise. [Hint: show that the set whose members are the natural numbers n such that every injective function from n to n is bijective is inductive.]

Corollary: No finite set is Dedekind infinite.

Proof: Exercise.

Corollary: If A is Dedekind infinite, then it is infinite.

Proof: Exercise.

Is the converse of this corollary true? We will return to this question later in this chapter.

Corollary: ω is infinite.

Proof: This follows from the fact, proved in Chapter 4, that the successor function is a bijection from ω to $\omega \setminus \{0\}$.

Corollary: No two distinct natural numbers are equinumerous.

Proof: Exercise. [Hint: use the fact (Chapter 4) that the \leq order on ω is connex.]

Corollary: For Any finite set A , there is a unique natural number equinumerous with A .

Proof: Exercise.

The unique natural number equinumerous with a finite set A is called the **cardinality** of A , written $|A|$.

Lemma: If $C \subsetneq n \in \omega$, then $C \approx m$ for some $m < n$.

Proof: Exercise. [Hint: show that the set whose members are those natural numbers n such that any proper subset of n is equinumerous to a member of n is inductive.]

Theorem: Any subset A of a finite set B is itself finite.

Proof: Let $n = |B|$, so there is a bijection $f: B \rightarrow n$. Then $f[A] \subseteq f[B] = n$. So either $f[A] = n$ or $f[A] \subsetneq n$. If $f[A] = n$, then $A \approx B$. If $f[A] \subsetneq n$, then by the previous lemma $f[A] \approx m$ for some $m < n$. \square

We say a set A is **dominated** by a set B , written $A \preceq B$, iff there is an injection from A to B , or, equivalently, iff A is equinumerous with a subset of B . If $A \preceq B$ and $A \not\approx B$, A is said to be **strictly dominated** by B , written $A \prec B$ or $A \prec\prec B$.

Some simple exercises are to show that for any sets A , B , and C , (a) $A \preceq A$; (b) if $A \preceq B$ and $B \preceq C$ then $A \preceq C$; and (c) $A \preceq \wp(A)$.

The Schröder-Bernstein Theorem: For any sets A and B , if $A \preceq B$ and $B \preceq A$, then $A \approx B$.

We have the resources to prove this, but since the proof is rather involved, we postpone it to the appendix.

Before continuing, we need to add to our list of assumptions about sets again (remember our last new assumption was that there is a set whose members are the natural numbers). To state the new assumption, we first need a couple of definitions. First, if A is a set, then the **nonempty power-set** of A , written $\wp_{ne}(A)$, is just $\wp(A) \setminus \{\emptyset\}$, i.e. the set of nonempty subsets of A . And second, a **choice function** for A is a function $c: \wp_{ne}(A) \rightarrow A$ such that, for each nonempty subset B of A , $c(B) \in B$. The new assumption is this:

Assumption of Choice: There is a choice function for any set.

It has been proved (by Paul Cohen, in 1963) that Choice is *independent* of the other assumptions we have made, in the sense that, if in fact our other assumptions are consistent, then either one of Choice or its denial (that some set does not have a choice function) can be added without leading to inconsistency. But as a practical matter, most working mathematicians prefer to assume Choice, because there are so many useful theorems that cannot be proved without it. One such theorem is the following:

Theorem: If A is infinite, then $\omega \preceq A$.

This proof is also deferred to the appendix.

The Dedekind-Pierce Theorem: A set is infinite iff it is Dedekind infinite.

Proof: The only-if part was proven above. Now suppose A is infinite. Then $\omega \preceq A$, that is, there is an injection $f: \omega \rightarrow A$. Now define a bijection $g: A \rightarrow A \setminus \{f(0)\}$ as follows: if $a \in A$ is not in the range of f , then $g(a) = a$; and if a is in the range of f , so that $a = f(n)$ for some $n \in \omega$, then $g(a) = f(n + 1)$. It is easy to see that g is injective and its range is $A \setminus \{f(0)\}$. \square

A set is said to be **countable** if it is dominated by ω . An infinite countable set is called **denumerable**, **denumerably infinite**, or **countably infinite**. A set which is not countable is called **uncountable**, **nondenumerable**, or **nondenumerably infinite**.

Corollary: Any countably infinite set is equinumerous with ω .

Corollary: Any infinite subset of ω is equinumerous with ω .

Some standard examples of countably infinite sets are the following: ω , $\omega \times \omega$, the positive natural numbers, the even natural numbers, \mathbb{Z} (the integers), and \mathbb{Q} (the rationals). Some standard examples of nondenumerable sets are \mathbb{R} (the reals), the subset I of \mathbb{R} consisting of the real numbers between 0 and 1 (including 0 and 1), and $\wp(\omega)$.

Now consider the following statement:

The Continuum Hypothesis: There is no set A such that $\omega \preceq A \preceq \wp(\omega)$.

This hypothesis has the same status as the Assumption of Choice: it can be proven to be independent of our other assumptions. The same is true of the following generalized form of the Continuum Hypothesis:

Generalized Continuum Hypothesis: For any infinite set B , there is no set A such that $B \preceq A \preceq \wp(B)$.

Appendix

The Schröder-Bernstein Theorem: For any sets A and B , if $A \preceq B$ and $B \preceq A$, then $A \approx B$.

Proof: By definition of \preceq , there are injections $f: A \rightarrow B$ and $g: B \rightarrow A$. Let C be the unique function from ω to $\wp(A)$ such that $C(0) = A \setminus \text{ran}(g)$ and $C(n+1) = g[f[C(n)]]$ for all $n \in \omega$; henceforth we write C_n for $C(n)$. Now we define $h: A \rightarrow B$ such that $h(x) = f(x)$ if $x \in \bigcup_{n \in \omega} C_n$ and $h(x) = g^{-1}(x)$ otherwise; this makes sense since $\text{ran}(g) = A \setminus C_0$. We will show h is bijective.

To show h is injective, suppose x and x' are distinct members of A ; it suffices to show that $h(x) \neq h(x')$. Since f and g^{-1} are one-to-one, we need only consider the case where $x \in C_m$ and $x' \notin \bigcup_{n \in \omega} C_n$. Now we define $D_n =_{\text{def}} f[C_n]$ for all $n \in \omega$, so that $C_{n+1} = g[D_n]$. Then $h(x) = f(x)$, which is in D_m ; but $h(x') = g^{-1}(x')$, which is *not* in D_m (since otherwise we would have $x' \in C_{m+1}$). So $h(x) \neq h(x')$, as desired.

To show h is surjective, let $y \in B$; we will show that $y \in \text{ran}(h)$. Clearly, for each n , $D_n \subseteq \text{ran}(h)$. So we can assume $y \in B \setminus \bigcup_{n \in \omega} D_n$. Next, we note that, for all n , $g(y) \notin C_n$ (the proof, which is inductive, is left as an exercise). Therefore $g(y) \notin \bigcup_{n \in \omega} C_n$. So $h(g(y)) = g^{-1}(g(y)) = y$. So $y \in \text{ran}(h)$. \square

Theorem: If A is infinite, then $\omega \preceq A$.

Proof: Let c be a choice function for A , and let h be the unique function from ω to $\wp(A)$ such that $h(0) = \emptyset$ and $h(n+1) = h(n) \cup \{c(A \setminus h(n))\}$ for all $n \in \omega$. Note for future reference that for any $m, n \in \omega$ with $m < n$, $h(m+1) \subseteq h(n)$. Also define $g: \omega \rightarrow A$ by $g(n) =_{\text{def}} c(A \setminus h(n))$, so that, for each $n \in \omega$, $h(n+1) = h(n) \cup \{g(n)\}$, and consequently also $g(n) \in h(n+1)$. Clearly, for all $n \in \omega$, $g(n) \notin h(n)$, since $g(n) = c(A \setminus h(n)) \in A \setminus h(n)$.

To complete the proof, we will show g is injective. So let m and n be distinct natural numbers; without loss of generality we can assume that $m < n$. Then $g(m) \in h(m+1)$, and so $g(m) \in h(n)$. But we already showed that $g(n) \notin h(n)$, so $g(m) \neq g(n)$; this shows g is injective as required. \square