Short Summary of Deductive Rules for Quantifiers

Notational conventions: “\( \phi(a) \)” indicates a formula with \( a \) occurring somewhere inside it (e.g. \( P(a) \), or \( (P(j) \rightarrow Q(a)) \), etc.). Then when “\( \phi(b) \)” is used afterward, it means the same formula as \( \phi(a) \) except that \( b \) appears where \( a \) appeared before.

Other logic textbooks use different symbols and different methods of indicating instantiations and generalizations in proofs; in this course, we will use only the system given in the textbook.

**Universal Instantiation (U.I., \( E \forall \)):** From \( \forall x \phi(x) \), derive \( \phi(v) \)

Two uses: (i) Where \( v \) is arbitrarily chosen: “Everything is mortal, so an arbitrarily chosen individual \( v \) is mortal.” (ii) Where a “real” constant is used instead of \( v \): “Everything is mortal, therefore Socrates in particular is mortal.”

Constraints on Rule: 1. Use \( v \) in U.I. (to indicate where U.G. can be used later). 2. The same \( v \) can be used in more than one instantiation. 3. You can use \( w \) instead of \( v \) if you want to use E.I. later rather than U.G. (subject to restrictions on \( w \) below), cf. example (7–31) p. 156.

**Universal Generalization (U.G., \( I \forall \)):** From \( \phi(v) \), derive \( \forall x \phi(x) \)

(\( v \) must be “arbitrarily chosen.”)

Constraints on Rule: Replace only \( v \), not \( w \).

**Existential Instantiation (E.I., \( E \exists \)):** From \( (\exists x) \phi(x) \) derive \( \phi(w) \)

“At least one thing is \( \phi \), therefore \( \phi \) is true of individual \( w \) (though we don’t know which individual \( w \) is).” (E.g. “There’s at least one cow in that barn. Let’s call her ‘Bossy’ for the moment.”)

Constraints on Rule: 1. Use \( w \) in E.I. (to indicate that an existential generalization, but not a universal generalization, can be used later). 2. If more that one application of E.I. occurs, a distinct \( w \) (\( w_1, w_2 \), etc.) must be used that does not appear earlier in the proof.

**Existential Generalization (U.G., \( I \exists \)):** From \( \phi(w) \), derive \( (\exists x) \phi(x) \)

E.G. Can also be used to replace a real constant, e.g. \( s \), as in “Socrates is a philosopher, therefore there is at least one philosopher”. Also, can replace \( v \) as well as \( w \).

Further General Constraints on Rules:

1. Instantiation rules can only be used to remove a quantifier that begins a formula (is the leftmost thing in the formula) and has widest scope (e.g. you can apply a rule to \( (\forall x)(P(x) \rightarrow Q(s)) \) but not to \( ((\forall x)P(x) \rightarrow Q(s)) \) or to \( ((\forall x)P(x) \rightarrow Q(s)) \)). Quantifier equivalences must be used to move the quantifiers to the beginning of the formula (Prenex Normal Form) before instantiation rules can be applied.

It also follows that in formulas with more than one quantifier, e.g. \( (\forall x)(\exists y)L(x, y) \), the left-most quantifier must be removed before the second one can be removed.
2. Similarly in replacing quantifiers, the added quantifier must go on the outside, e.g. generalizing \( (\exists x)(L(x, v)) \) gives \( ((\forall y)(\exists x)(L(x, y)), \) not \( ((\exists x)(\forall y)(L(x, y)).\)

3. \( v \) or \( w \) can be introduced in the auxiliary premise of a conditional proof and generalized later (see p. 158).

4. In applying generalizations, care must be taken to ensure that distinct variables do not become confused (certain exceptions allowed): see top of p. 159 for details and examples.

5. When two existential generalizations occur, or two universal generalizations occur, the order in which they are applied does not matter. But if both U.G. and E.G. occur, then their order is constrained by the order in which U.E. and E.I. introduced \( v \) and \( w \); specifically, if U.I. came before E.I., then E.G. must be applied before U.G. (But if E.I. came first, then either order of generalization is OK.) One way to remember these is to note that these constraints would allow you to derive \( (\forall y)(\forall x)\psi \) from \( (\forall x)(\forall y)\phi \); to derive \( (\exists y)(\exists x)\psi \) from \( (\exists x)(\exists y)\phi \); to derive \( (\forall x)(\exists y)\psi \) from either \( (\forall x)(\exists y)\phi \) or from \( (\exists y)(\forall x)\phi \), but prohibit you from deriving \( (\exists y)(\forall x)\psi \) from \( (\forall x)(\exists y)\phi \). If you will recall the Quantifier Exchange laws (p. xxx), you can see why these constraints should hold.

6. In an auxiliary premise you can use \( v_1, v_2 \), then after you withdraw the auxiliary premise, you U.G. over the \( v_1 \). This saves the step of using \( (\forall x)(\ldots \text{in the aux. prem. and then instantiating it with } v_1 \ldots) \), and it has other advantages in certain proofs. This corresponds to the informal proof step where you say “Suppose \( v \) is an arbitrarily chosen such-and-such,”.)