Deductive Systems

What is the difference between a deductive system and the set of rules of inference used to define the system?

A **deductive system** consists of all the possible deductions (or, all derivations) \( \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \) that can be constructed using these rules (correctly). Note that a deductive system, defined this way, is an infinite set of deductions. The relationship between the rules of inference and the deductive system as a whole is somewhat like that between a recursive definition and the set it defines: the rules of inference give you a means of constructing new derivations out of ones you already have, and the process can be repeated as long as you want.

Sometimes, a rule of inference can be derived out of other rules of inference in the initially given rule set: we saw an example of this when we derived the rule **Hypothetical Syllogism** (i.e. \( (\phi \rightarrow \psi), (\psi \rightarrow \chi) \vdash (\phi \rightarrow \chi) \)) from \( E \rightarrow \) (modus ponens) and \( I \rightarrow \) (conditional proof).

In other words, two or more different sets of rules of inference can define the same deductive system. (When a rule (or axiom) has this “redundancy” (as Hypothetical Syllogism does in our rules for statement logic), it is said that the rule is **not independent** of the other rules.)

On the other hand, if you start with too few rules of inference, the resulting deductive system produced may not include some of the theorems that are desired.

Therefore, it is an interesting (and often complicated) question just which combinations of a few rules of inference will enable you to produce all the desired derivations and which will not. There are a number of different combinations that can produce the full deductive system known as statement logic.

To illustrate the difference between a small but adequate set of rules for the statement logic deductive system, versus an inadequate set, the textbook by Gamut (in the reading on reserve) introduces the rules in a way that shows how proper subsets of the rules will fail to give you the full deductive system of statement logic: this is the system on the handout “An alternative formulation . . . ” In other words, these smaller lists of rules each define a deductive system that is a proper subset of the standard statement logic system. Here are the systems that will result:

<table>
<thead>
<tr>
<th>Set of rules:</th>
<th>Name of System:</th>
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<tbody>
<tr>
<td>( \land I, \land E; \lor I, \lor E; \rightarrow I, \rightarrow E; \neg I, \neg I ), i.e. rules 1–7 on the following page</td>
<td>Minimal Logic (p. 139)</td>
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<tr>
<td>Minimal Logic plus EFSQ (rule 8)</td>
<td>Intuitionist Logic (p. 140)</td>
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<tr>
<td>Intuitionist Logic plus Double Negation (rule 9)</td>
<td>Classical Propositional Logic</td>
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What are the differences between the three systems?

1. As Gamut notes (top of p. 139), in minimal logic we cannot prove the so-called **Law of the Excluded Middle**, \( \vdash (p \lor \neg p) \), or the EFSQ inference.

2. In Intuitionist Logic, we can prove some further derivations, including \( (p \lor q), \neg q \vdash p \), but still not Excluded Middle.
3. Only in the Classical System can we derive the full set of inferences we want.

(By setting up rules to make this “demonstration” possible, Gamut had to choose Inference rules for $\neg$ and $\lor$ that would not have been the easiest ones to use if they had only been interested in a set of rules for Classical Logic.)