Formal Deductive Systems and Model Theory

This handout is intended as an overview and omits a number of details: For more information, see the discussion of inference in deduction vs. entailment in semantics and of meta-theorems of logic in MMiL pp. 199-201 or GAMUT, pp. 148-155 (on reserve). A warning is hereby issued that terminology is not exactly the same in all works on logic, but the ideas here are central to the study of logic.

1 Deductive Systems

Formal deductive systems give a SYNTACTIC characterization of a logical system (and of other formal theories, as we’ll see later), i.e. such systems involve only relationships among symbolic expressions, including formulas, determined by their syntactic structure. Model theory (often referred to as formal semantics by linguists) gives a SEMANTIC characterization of a logical system, since it involves relations between logical expressions and the objects they denote and the facts they express.

A deductive system for a logic is a kind of AXIOMATIC SYSTEM, which we’ll talk about soon (cf. MMiL p. 182-190). There are two varieties of such systems, NATURAL DEDUCTION SYSTEMS, and AXIOMATIZATIONS. The former was the kind we used in this course for both Statement Logic and 1st-Order Logic; (for the latter, see MMiL. pp. 220-222.)

1.1 Syntax and Deduction

1. We must have an explicit SYNTAX, a syntactic definition of WELL-FORMED-FORMULA (or WFFs). Of course, there are many different systems of logic—statement logic, predicate logic, modal logics of several kinds, etc.—and our syntax as well as our deductive rules will depend on which one we are talking about. The set of wffs is normally specified by a recursive definition.

We may also adopt definitions of some expressions in terms of others. For example, we might omit \( \rightarrow \) from the syntactic rules and then introduce it via a definition in terms of \( \lor \) and \( \neg \). Thus \( p \rightarrow q \) is officially simply an abbreviation for \( \neg p \lor q \).

2. AXIOMS. (Axiomatic systems in general have both axioms and rules of inference: Natural Deduction systems differ from other axiomatic systems in having no axioms, so we’ll ignore axioms for now and come back to them later in this course).

3. There is a set of RULES OF INFERENCE. For statement logic (in the natural deduction version), these will include Modus Ponens (to infer \( \psi \) from \( \phi \) and \( \phi \rightarrow \psi \)), modus tollens, simplification, etc. (Equivalently, \( E\land, I\land, E\lor \), etc.)

4. A DERIVATION, or PROOF, of a wff \( \phi \) from a set of premises \( \Gamma \) is a finite sequence of lines, each consisting of a wff, such that \( \phi \) is the last line and each line meets one of the following conditions:

(i.) (Any axiom may appear on a line) (ii.) Any premise (= member of \( \Gamma \)) may appear on a line, (iii.) Any wff may appear as a line if it follows from previous line(s) by one of the inference rules.

The assertion that there is a derivation of \( \phi \) from a set of wffs \( \Gamma \) (premises) is symbolized \( \Gamma \vdash \phi \). We also say that \( \phi \) follows from \( \Gamma \) in the system, or that \( \Gamma \) logically implies \( \phi \).
4. A wff is called a theorem of the deductive system if and only if it is derivable from the empty set of premises; this is symbolized $\vdash \phi$. (Theorems were the same as tautologies in MMIL, but we use ‘theorem’ here to avoid a possible ambiguity.)

5. A wff is a contradiction iff its negation is a theorem. (The symbol $\perp$ is often used to stand for any contradiction; similarly, $\top$ stands for any theorem.)

6. Two wffs $\phi$ and $\psi$ are deductively equivalent (or interderivable, or logically equivalent) if and only if both $\{ \phi \} \vdash \psi$ and $\{ \psi \} \vdash \phi$.

2 Semantics, or Model Theory

Note that a deductive system, as defined above, is described only in terms of syntactic properties of formulas—we made no mention of truth values, denotations, etc. Semantics (as the term is used in logic), on the other hand, describes relationships between formulas and things which the formulas are about (i.e. truth values, denotations); these are the interpretations of formulas.

When we begin to talk about semantics, it is essential to keep in mind the distinction between object language and metalanguage. The object language is the formal language we are trying to characterize, e.g. statement logic or predicate logic. A theorem of the object language is one that uses only the syntactic properties of that “self-contained” system (including rules for deductions). The metalanguage is the language used to talk about object language expressions (names, formulas, predicates, connectives). We have used English as our metalanguage; in the meta-language, we mention but do not use object language expressions.

7. A model for predicate logic (symbolized $M$) consists of a domain of discourse (often symbolized $D$), which can be any non-empty set, and an interpretation function (or valuation), symbolized $F_M$, assigns a denotation to each constant of the language. The model is then the ordered pair $(D, F)$. (For statement logic, the constants are the so-called atomic propositional "variables", $p, q, r, \ldots$ and they are assigned a truth value by what we called a valuation $V$.) For predicate logic, of course, there are other types of constants, and variables get values assigned by a different function, an assignment of values to variables, or simply called a variable assignment; for propositional logic, there is no domain of discourse as such other than possible assignments of truth values).

8. The semantic rules specify, in a recursive fashion, how the denotation (truth value) of a complex expression (complex wff) can be determined from the syntactic steps by which it was built up, given a particular interpretation $F$ and domain $D$. See pp. 315–331 in the text for a fully formalized semantics for statement and predicate logic.

9. A wff $\phi$ is contingent iff it is true in some models and false in others.

10. A wff $\phi$ is valid (sometimes called universally valid (a tautology) iff it is true in all models. This is symbolized $\models \phi$. (Note the difference between a valid proof and a valid formula.)

11. A wff $\phi$ is a (semantic) contradiction iff it is false in all models. This can be symbolized $\models \neg \phi$.

12. A set of wffs $\Gamma$ semantically entails a wff $\phi$ (or, $\phi$ is a semantic consequence of $\Gamma$) iff every model in which all the formulas in $\Gamma$ are true is also a model in which $\phi$ is true. This is symbolized $\Gamma \models \psi$.

14. Two formulas are semantically equivalent iff they are true in exactly the same models.
The most common way to symbolize this is $\phi \equiv \psi$.

15. Alternatively we can say that two formulas are semantically equivalent iff each entails the other.

3 Logical Metatheory

Logical metatheory means the study of properties of a logical system, both properties of the deductive system by itself and relationships between a logical system and its model theory. "Metatheorems" are theorems about a logical system, as contrasted with theorems in a logical system (i.e., formulas in the object language that can be proved with no premises).

1. A formal deductive system is decidable if there exists a procedure for determining in a finite number of steps whether or not a given wff is a theorem or not (and likewise, whether an argument is valid).

   For example, statement logic is decidable because the procedure of simply constructing a truth table for a formula will always show whether it is a theorem, and this can be done for any formula whatsoever. Note that attempting to find a proof of the formula is not a decision procedure, because if we fail to find a proof this could either be because the formula is not a theorem or simply because we have not been ingenious enough. Predicate logic with only one-place predicates (Monadic predicate logic) is also decidable, but it turns out that predicate logic with two or more-place predicates is not decidable. See the GAMUT reading for a more complete discussion.

2. A formal deductive system is consistent iff it is never the case that both a formula and its negation can both be derived as theorems. (Note that an inconsistent system could not have a model, since it is not possible for a formula to be both true and false in the same model, so one way of showing that a system is consistent is by showing that it has a model.)

4. An axiom and/or a rule of inference is independent of the other axioms if it cannot be omitted from the system without making it impossible to derive certain theorems; if its omission does not result in the loss of any theorems, then it is not independent. Equivalently, if an axiom/rule is independent, then it can’t be derived from the other axioms/rules. For example, we showed that the rule of Hypothetical Syllogism is not independent by deriving it as a theorem. In the alternative formulation of statement logic on the handout (the rules are called $\land$-E, $\land$-I, etc), all rules of inference are independent.

5. A deductive system is deductively complete (or complete with respect to negation, or syntactically complete, or formally complete) iff for every wff $\phi$, either $\phi$ or $\neg \phi$ can be proved. Many important logics are not deductively complete, of course: Statement logic and 1st-order logic are not. (Don’t confuse deductively complete with semantic completeness, cf. below.)

The above properties are properties of the deductive system itself, not the relationship of it to its interpretation (though some can be most easily proved by reference to an interpretation.) The most important are decidability and consistency. Those below do essentially involve the relationship between deduction and interpretation:

4. A formal deductive system is sound iff all its theorems are valid formulas; in symbols, iff it is the case that:

   $$\text{for all } \phi, \text{ if } \vdash \phi, \text{ then } \models \phi.$$  

   (Example: classical logic is sound, with respect to its standard semantic interpretation, which we used in this course.)
5. A formal deductive system is (semantically) complete (or complete with respect to an interpretation, or weakly semantically complete) iff every semantically valid formula can be deduced as a theorem, i.e.:

for all $\phi$, if $|$ $\models$ $\phi$, then $\vdash$ $\phi$.

(Example: Classical propositional logic is complete with respect to its standard interpretation (the truth tables), but intuitionist logic\(^1\) would not be complete with respect to that interpretation, although it is complete with respect to a different kind of interpretation (see textbook if you’re interested).

(As Rich Thomason wrote “Completeness means you can prove enough, soundness means can’t prove too much.”)

Note that to say that a system is both sound and complete (with respect to its interpretation) is to say that the semantic and the deductive characterizations of validity (theoremhood) coincide, as do the related notions of entailment (cf. provability), semantic equivalence (cf. indeducibility), etc.

for all $\phi$, $|$ $\models$ $\phi$ if and only if $\vdash$ $\phi$.

(Actually, there are various slightly different notions of semantic completeness; the above is the most common and important.) We have in effect been taking it for granted that statement logic and 1st-order logic are sound and complete, because we have been constantly switching back and forth between deductive methods and semantic methods as we learned them. However, the actual proofs of soundness and completeness are not at all trivial: see the GAMUT reading on reserve for more details.

Not all these properties are equally important: the most important are consistency, soundness, and completeness. However, many logics of interest are not complete: second order logic (a logic which has variables over predicates as well as variables over individuals) are not complete, nor are higher-order logics. For example, Montague’s intensional ”logic” (in PTQ) is a higher-order logic and is therefore not complete. (It is more properly called a formal interpreted language than a logic, since Montague gave syntactic rules and a formal interpretation but no deductive rules.) The definition of entailment that such systems yield semantically can nevertheless be examined precisely for many purposes (e.g. natural language semantics).

Another comment: Even though a deductive system can fail to be complete with respect to one kind of semantic interpretation, it could still be complete with respect to a different kind of semantic interpretation. Though intuitionist logic is incomplete in the standard interpretation (based on truth functions and models as we defined them), it is possible to define a different interpretation for which it is complete. Other logics (modal logics, higher-order logic) also need different kinds of interpretation systems than standard logic does.

\(^1\)Intuitionist logic is a system like classical logic except that the law of double negation does not hold, i.e. $\neg\neg\phi$ is not equivalent to $\phi$ ($\phi$ $\vdash$ $\neg\neg\phi$ but not the other way around. Intuitionist logic consists of the first eight rules on the alternative formulation handout without the ninth rule, Double Negation.)
4 Object Language vs. Meta-Language in proofs about logical systems

Notice that in stating (and proving) meta-theorems, we use a meta-language, we don’t state them within statement logic or predicate logic itself; in the meta-language we refer to (mention) formulas of the object language, but we don’t use them to communicate with (in the sense of “referring” vs. “use”. Note that the symbols ⇒, ⇐, ⊢, ≡, | and [ ] are part of the metalanguage, not the object language.

Still, we assume that valid logical principles apply, implicitly, to our meta-language proofs stated in English (or quasi-logical English). Indeed, all proofs in mathematics, including mathematical linguistics, are intended to be valid proofs by the rigorous standards of classical propositional and predicate logic, although stated in a less formalized way. In other words, it’s assumed that the author of a proof could, if questioned, give a formalized version of it that is fully correct technically. See the section of the book on “Formal vs. Informal Proofs” for more explanation of this.

Nevertheless, we cannot literally state or prove metatheorems about a formal language by using that language simultaneously as both object language and metalanguage. The reason for this is that semantic paradoxes will arise when we use a language and refer to sentences within that language at the same time, called “paradoxes of self-reference”: i.e. a sentence can be proved to both true and false, hence the reasoning is inconsistent. (Cf. the ‘Paradox of the Liar’ below). Consequently, proofs about a formal language (and for that matter, also the definitions of models and semantic rules used to define “true sentence”) must always be stated in an independent metalanguage, for which the semantics is implicitly known already. (We could formally describe the semantics of the meta-language itself by using still another language to define it in, a “meta-meta-language”, and so one. But at each step, the meta-language and the object language must be distinct. (cf. however, recent research in logical metatheory by Saul Kripke and by Aneal Gupta.) Here are two ways to see the paradox:

(1) This sentence is false.

(2) a. Sentence (2b) is true.
   b. Sentence (2a) is false.

Such paradoxes will arise in a language just in case it has (i) a means for referring to sentences within the language itself, and (ii) a predicate “is true” (and/or “is false”). Natural languages are often considered to be somehow different from formal languages in that we both use English (for example) and refer to English sentences at the same time, and can assert that sentences are true or false. Normally we do not do both at once, however, and the consequences of this observation for the semantics of natural languages is unclear.)