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MATH 363
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6.1.5)

a) We are given $f(x;\theta) = \frac{xe^{-\frac{x}{\theta}}}{\theta^2}$, so

$$\begin{aligned}
L(x;\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \theta^{-2n} \left(\prod_{i=1}^{n} x_i \right) \left(e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}} \right) \\
\ln L(x;\theta) &= -2n \ln\theta + \sum_{i=1}^{n} \ln x_i - \frac{\sum_{i=1}^{n} x_i}{\theta} \\
\frac{\partial \ln L(x;\theta)}{\partial\theta} &= -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2}
\end{aligned}$$

Solving this for zero yields

$$\begin{aligned}
\frac{2n}{\hat\theta} &= \frac{\sum_{i=1}^{n} x_i}{\hat\theta^2} \\
2n\hat\theta &= \sum_{i=1}^{n} x_i \\
\hat\theta = \frac{\sum_{i=1}^{n} x_i}{2n} &= \frac{\bar{x}}{2}
\end{aligned}$$

b) We are given $f(x;\theta) = \frac{x^2e^{-\frac{x}{\theta}}}{2\theta^3}$, so

$$\begin{aligned}
L(x;\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \frac{\theta^{-3n}}{2} \left(\prod_{i=1}^{n} x_i^2 \right) \left(e^{-\frac{\sum_{i=1}^{n} x_i}{\theta}} \right) \\
\ln L(x;\theta) &= -3n \ln 2\theta + \sum_{i=1}^{n} 2\ln x_i - \frac{\sum_{i=1}^{n} x_i}{\theta} \\
\frac{\partial \ln L(x;\theta)}{\partial\theta} &= -\frac{6n}{2\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2}
\end{aligned}$$

Solving this for zero yields

$$\begin{aligned}
\frac{3n}{\hat\theta} &= \frac{\sum_{i=1}^{n} x_i}{\hat\theta^2} \\
3n\hat\theta &= \sum_{i=1}^{n} x_i \\
\hat\theta = \frac{\sum_{i=1}^{n} x_i}{3n} &= \frac{\bar{x}}{3}
\end{aligned}$$

c) We are given $f(x;\theta) = \frac{e^{-|x-\theta|}}{2}$, so

$$\begin{aligned}
L(x;\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \frac{e^{-\sum_{i=1}^{n}|x_i-\theta|}}{2^n} \\
\ln L(x;\theta) &= -n\ln 2 - \sum_{i=1}^{n}|x_i-\theta| \\
\frac{\partial \ln L(x;\theta)}{\partial\theta} &= \sum_{i=1}^{n}\frac{|x_i-\theta|}{x_i-\theta}
\end{aligned}$$

Since this is the sum of a function that can only take the values 1 and -1, it will be zero when exactly half of the $x_i - \hat{\theta}$ terms are 1 and half are -1, or when half of the x_i terms are greater than $\hat{\theta}$ and half less than $\hat{\theta}$. This occurs when $\hat{\theta}$ is the sample median.

6.1.7)

a)

b) We are given $f(x;\theta) = \theta x^{\theta-1}$, so

$$\begin{aligned}
L(x;\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \theta^n \prod_{i=1}^{n} x_i^{\theta-1} \\
\ln L(x;\theta) &= n\ln\theta + \sum_{i=1}^{n}(\theta-1)\ln x_i \\
\frac{\partial \ln L(x;\theta)}{\partial\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n}\ln x
\end{aligned}$$

Solving this for zero yields

$$\begin{aligned}
-\frac{n}{\hat{\theta}} &= \sum_{i=1}^{n}\ln x_i \\
\hat{\theta} &= \frac{-n}{\ln\prod_{i=1}^{n} x_i}
\end{aligned}$$

c) From the example on page 289, we know that the method-of-moments estimator $\tilde{\theta}$ is $\frac{\bar{x}}{1-\bar{x}}$. For the first set, $\hat{\theta} = 0.5493$ and $\tilde{\theta} = 0.5975$; for the second set, $\hat{\theta} = 2.2101$ and $\tilde{\theta} = 2.4004$; for the third set, $\hat{\theta} = 0.9586$ and $\tilde{\theta} = 0.8646$.

d)

6.1.8)

a) We are given $f(x;\theta) = \frac{x^{\frac{1-\theta}{\theta}}}{\theta}$, so

$$\begin{aligned}
L(x;\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \frac{\prod_{i=1}^{n} x_i^{\frac{1-\theta}{\theta}}}{\theta^n} \\
\ln L(x;\theta) &= -n\ln\theta + \sum_{i=1}^{n}\left(\frac{\ln x_i}{\theta} - \ln x_i\right) \\
\frac{\partial \ln L(x;\theta)}{\partial\theta} &= -\frac{n}{\theta} + \sum_{i=1}^{n} -\frac{\ln x_i}{\theta^2}
\end{aligned}$$

Solving this for zero yields

$$\begin{aligned}
\frac{n}{\hat{\theta}} &= -\frac{\sum_{i=1}^{n} \ln x_i}{\theta^2} \\
\hat{\theta} &= -\frac{\sum_{i=1}^{n} \ln x_i}{n}
\end{aligned}$$

b) We have $E(\hat{\theta}) = -\frac{E\left(\sum_{i=1}^{n} \ln x_i\right)}{n}$. Since $E\left(\sum_{i=1}^{n} \ln x_i\right) = -n\theta$, this expression equals $\frac{n\theta}{n} = \theta$. Since $E(\hat{\theta}) = \theta$, the estimator is unbiased.

6.1.20)

This is a hypergeometric distribution with $n = 8$, $N = 64$, $N_2 = 64 - N_1$, and parameter $\theta = N_1$. For such a distribution, $\mu = n\frac{N_1}{N} = \frac{\theta}{8}$. By the method of moments, then, we can use $\tilde{\theta} = 8\bar{x}$ as an estimator for θ. Then $\bar{x} = 1.467$ and $\tilde{\theta} = 11.733$. We therefore guess that there are 12 orange balls and 52 blue balls in the urn.

6.2.6)

The confidence interval is given by

$$\bar{x} \pm t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}} = 11.95 \pm \frac{(1.96)(11.8)}{\sqrt{37}} = (8.148, 15.752)$$

6.2.14)

The confidence interval is given by

$$\bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}} = 6.05 \pm \frac{(2.576)(0.02)}{\sqrt{1219}} = (6.0485, 6.0515)$$

6.3.2)

Here, the variances of the populations are equal. We have $n = 5$, $\bar{x} = 539.2$, $s_x = 62.92$, $m = 8$, $\bar{y} = 544.625$, and $s_y = 61.538$. The confidence interval is given by

$$(\bar{x} - \bar{y}) \pm t_{\alpha/2}(m+n-2)(s_p)\sqrt{\frac{1}{n} + \frac{1}{m}}$$

Since

$$s_p = \sqrt{\frac{(n-1)s_x^2 + (m-1)s_y^2}{m+n-2}} = 62.044$$

the interval is

$$-5.425 \pm (1.796)(62.044)(0.5701) = (-68.95, 58.100)$$

6.3.4)

Here, the variances of the populations are not equal. We have $n = 7$, $\bar{x} = 1511.714$, $s_x = 206.335$, $m = 10$, $\bar{y} = 1118.4$, and $s_y = 117.336$.

a) The point estimate is $\bar{x} - \bar{y} = 393.314$.

b) The confidence interval is given by

$$(\bar{x} - \bar{y}) \pm t_{\alpha/2}(m+n-2)\sqrt{\frac{s_x^2}{n} + \frac{s_y^2}{m}} = 393.314 \pm (2.131)(78.062) = (226.963, 559.665)$$

6.3.8)

a) The point estimate is $\bar{x} = 0.07875$.

b) The confidence interval is given by

$$\bar{x} \pm t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}} = 0.07875 \pm \frac{(2.069)(0.2496)}{\sqrt{24}} = (-0.02665, 0.1842)$$

c) Since the confidence interval is centered around a positive number, we can say that the program was probably effective.

6.5.4)

Let $\hat{p} = \frac{y}{n} = 0.700$. Then the desired interval is given by

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.7 \pm 1.96\sqrt{\frac{(0.7)(0.3)}{1234}} = (0.6744, 0.7256)$$

6.5.18)

a) Let $\hat{p}_A = \frac{y_A}{n_A} = 0.369$ and $\hat{p}_B = \frac{y_B}{n_B} = 0.320$. Then the desired interval is given by

$$\hat{p}_A - \hat{p}_B \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_A(1-\hat{p}_A)}{n_A} + \frac{\hat{p}_B(1-\hat{p}_B)}{n_B}} = 0.04911 \pm 1.96\sqrt{0.0005062 + 0.0004945} = (-0.01289, 0.1111)$$

b) If the two versions were consistent, we would expect the interval to be centered around zero, as an ideal difference would equal zero. Since this interval is skewed toward the right, the two versions are not consistent.

Handout #7)

We are given $f(x) = \frac{x}{\theta}e^{-\frac{x^2}{2\theta}}$, so

$$\begin{aligned}
E(Y) &= \int_0^\infty x^2 \frac{x}{\theta} e^{-\frac{x^2}{2\theta}}\,dx \\
&= -x^2 e^{-\frac{x^2}{2\theta}} + \int_0^\infty 2x e^{-\frac{x^2}{2\theta}}\,dx \\
&= \left[-x^2 e^{-\frac{x^2}{2\theta}} - 2\theta e^{-\frac{x^2}{2\theta}}\right]_0^\infty \\
&= \left[-e^{-\frac{x^2}{2\theta}}(2\theta + x^2)\right]_0^\infty \\
&= 2\theta
\end{aligned}$$

$Var(Y) = E(Y^2) - (E(Y))^2 = E(X^4) - 4\theta^2$. We note that

$$\begin{aligned}
E(x^4) &= \int_0^\infty x^4 e^{-\frac{x^2}{2\theta}}\,dx \\
&= -x^4 e^{-\frac{x^2}{2\theta}} + \int_0^\infty 4x^3 e^{-\frac{x^2}{2\theta}}\,dx \\
&= -x^4 e^{-\frac{x^2}{2\theta}} - 4\theta x^2 e^{-\frac{x^2}{2\theta}} + \int_0^\infty 8\theta x e^{-\frac{x^2}{2\theta}}\,dx \\
&= \left[-x^4 e^{-\frac{x^2}{2\theta}} - 4\theta x^2 e^{-\frac{x^2}{2\theta}} - 8\theta^2 e^{-\frac{x^2}{2\theta}}\right]_0^\infty \\
&= \left[-e^{-\frac{x^2}{2\theta}}(x^4 + 4\theta x^2 + 8\theta^2)\right]_0^\infty \\
&= 8\theta^2
\end{aligned}$$

and therefore $Var(Y) = 8\theta^2 - 4\theta^2 = 4\theta^2$.

Handout #8)

a) We are given $f(x) = \frac{x}{\theta}e^{-\frac{x^2}{2\theta}}$, so

$$\begin{aligned} L(x;\theta) &= \prod_{i=1}^{n} f(x_i;\theta) \\ &= \frac{\prod_{i=1}^{n} x_i}{\theta^n} e^{-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}} \end{aligned}$$

b)

$$\begin{aligned} \ln L(x;\theta) &= \ln \prod_{i=1}^{n} x_i - \ln(\theta^n) + \sum_{i=1}^{n} -\frac{x_i^2}{2\theta} \\ &= \sum_{i=1}^{n} \ln x_i - n\ln\theta - \frac{\sum_{i=1}^{n} x_i^2}{2\theta} \\ \frac{\partial \ln L(x;\theta)}{\partial\theta} &= -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} \end{aligned}$$

Solving this for zero yields

$$\begin{aligned} \frac{\sum_{i=1}^{n} x_i^2}{2\hat{\theta}^2} &= \frac{n}{\hat{\theta}} \\ \hat{\theta} &= \frac{\sum_{i=1}^{n} x_i^2}{2n} \end{aligned}$$

c)

$$\begin{aligned} E(\hat{\theta}) &= \frac{E(\sum_{i=1}^{n} x_i^2)}{2n} \\ &= \frac{nE(X^2)}{2n} \\ &= \frac{2\theta}{2} = \theta \end{aligned}$$

Since $E(\hat{\theta}) = \theta$, $\hat{\theta}$ is an unbiased estimator of θ.

d)

$$\begin{aligned} Var(\hat{\theta}) &= \frac{Var(\sum_{i=1}^{n} x_i^2)}{(2n)^2} \\ &= \frac{n(Var(X^2))}{4n^2} \\ &= \frac{4\theta^2}{4n} \\ &= \frac{\theta^2}{n} \end{aligned}$$

$$SE(\hat{\theta}) = \frac{SD(\hat{\theta})}{\sqrt{n}} = \frac{\theta}{\sqrt{n}}\sqrt{n} = \theta$$

e) Since $\hat{\theta}$ is unbiased, $b(\hat{\theta}) = 0$ for all n. We note that $Var(\hat{\theta}) = \frac{\hat{\theta}^2}{n} \to 0$ as $n \to \infty$. Therefore $\hat{\theta}$ is a consistent estimate of θ.

f) The 95% confidence interval is given by $\bar{x} \pm \frac{(1.96)\sqrt{Var(\theta)}}{\sqrt{n}} = \bar{x} \pm 1.96\theta$.