Complete Partial Orders, PCF, and Control

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Abstract

We develop the theory of directed complete partial orders and complete partial orders. We review the syntax of the abstract programming language PCF, and present its interpretation into the cartesian closed category of complete partial orders. We then present category theoretic interpretations for control operators. This material is mostly drawn from Chapters 1, 6, and 8 in Amadio & Curien (1998).

Introduction

Just a few words about where this is going. We want to understand continuations, so we have determined the following path to understanding them:

Continuations

The syntax and semantics of control operators

The syntax and semantics of PCF

Cpos as CCCs   Syntax and semantics of λ-theories

We assume familiarity with the syntax and semantics of λ-theories, thus it will be covered only in passing. We assume the reader has some exposure to category theory, though we will present enough of it for readers without extensive experience. Thus, we begin with the category theory of cpos (Section 1), and work toward the syntax and semantics of PCF (Section 2). We then present Scott topologies and the category of dcpos with partial continuous functions as arrows (Section 3). We then present the syntax and semantics of control operators (Section 4).

1 Directed Complete Partial Orders

1.1 Definitions and Examples

Definition 1. Given a partial order $(D, \leq)$, a non-empty subset $\Delta \subseteq D$ is called *directed* if, for all $x, y \in \Delta$, there is a $z \in \Delta$ such that $x \leq z$ and $y \leq z$.

We write $\Delta \subseteq_{dir} D$ if Δ is a directed subset of D.

Example 1. The natural numbers $\mathbb{N}$ with the usual $\leq$ relation is a partial order $(\mathbb{N}, \leq)$. Let Δ be any subset of $\mathbb{N}$, and consider $x, y \in \Delta$. It is easy to see that $z = \max(x, y) \in \Delta$ is such that $x \leq z$ and $y \leq z$. That is, $\Delta \subseteq_{dir} \mathbb{N}$.

Definition 2. A partial order $(D, \leq)$ is called a *directed complete partial order* (dcpo) if each $\Delta \subseteq_{dir} D$ has a least upper bound (lub), denoted $\bigvee \Delta$. A directed complete partial order that has a least element, denoted $\perp$, is called a *complete partial order* (cpo).

Example 2. Let $\mathbb{N}_n$ be the set of the first n natural numbers. Then $(\mathbb{N}_n, \leq)$ is a dcpo. Moreover, since $0 \leq k$ for all $k \in \mathbb{N}_n$, $(\mathbb{N}_n, \leq)$ is a cpo with least element 0. Notice that $(\mathbb{N}, \leq)$ is not a dcpo, since the set of odd natural numbers has no least upper bound.

Example 3. Let $\mathbb{Z}_n$ be the set of integers k such that $k \leq n$. Then $(\mathbb{Z}_n, \leq)$ is a dcpo. Notice that $(\mathbb{Z}_n, \leq)$ is not a cpo, since it has no least element.

Let $(D, \leq)$ be a partial order. An *infinite ascending chain* is a sequence $x_0, x_1, \ldots, x_n, \ldots$ of distinct elements in D such that $x_0 \leq x_1 \leq \cdots \leq x_n \cdots$. The unbounded sets $\mathbb{N}$ and $\mathbb{Z}$ are not suitable for forming dcpos, as they contain infinite ascending chains (e.g. the set of odd natural numbers). The upper bounds on their counterparts $\mathbb{N}_n$ and $\mathbb{Z}_n$ disallow infinite ascending chains, making both $\mathbb{N}_n$ and $\mathbb{Z}_n$ suitable for dcpos. We can generalize this observation.

Example 4. All partial orders without infinite ascending chains are dcpos.

The following example provides a method for constructing a cpo out of any given set.

Example 5. Let X be any set. Define $X_\perp = X \cup \{\perp\}$ where $\perp \notin X$, and for $x, y \in X_\perp$, define $x \leq y$ iff $x = \perp$ or $x = y$. Then $(X_\perp, \leq)$ is a cpo.

A cpo $X_\perp$ constructed this way is called *flat*, due to the appearance of its hasse diagram. The following examples of flat cpos will be important in later developments.

Example 6. Let $\mathbf{B} = \{tt, ff\}$, where tt and ff are called *truth values*. Let ω be the smallest infinite limit ordinal. Then both $\mathbf{B}_\perp$ and $\omega_\perp$ are flat cpos.

Definition 3. Let $(D, \leq)$ and $(D', \leq')$ be partial orders. A function $f : D \to D'$ is called *monotonic* if, for all $x, y \in D$, if $x \leq y$, then $f(x) \leq' f(y)$.

Example 7. Let $f : \mathbb{N} \to \mathbb{N}$ be the function $f(n) = n + 1$. Then f is monotonic, given the partial order $(\mathbb{N}, \leq)$.

Example 8. Let $f : \mathbb{N}_n \to \mathbb{N}_{n+1}$ be defined as above. Then f is monotonic, given the partial orders $(\mathbb{N}_n, \leq)$ and $(\mathbb{N}_{n+1}, \leq)$.

Definition 4. Let $(D, \leq)$ and $(D', \leq')$ be dcpos. A function $f : D \to D'$ is called *continuous* if f is monotonic and, for all $\Delta \subseteq_{dir} D$, $f(\bigvee \Delta) = \bigvee f(\Delta)$[1].

When the second condition in the definition is satisfied, we say that f preserves directed lubs.

Example 9. Consider $(\mathbb{N}_n, \leq)$ and $(\mathbb{N}_{n+1}, \leq)$, and $f : \mathbb{N}_n \to \mathbb{N}_{n+1}$ defined in Example 7. We want to show that f is continuous. We have already established that f is monotonic. We just need to show that f preserves directed lubs. Let $\Delta \subseteq_{dir} \mathbb{N}_n$. Then $\bigvee \Delta$ is simply the largest natural number

[1] $f(\Delta) = \{f(\delta) \mid \delta \in \Delta\}$.

in Δ, call it δ. Since $k \leq \delta$ for all $k \in \Delta$, it follows from the monotonicity of f that $f(k) \leq f(\delta)$ for all $k \in \Delta$. That is, $f(\delta)$ is the largest natural number in $f(\Delta)$, hence $\bigvee f(\Delta) = f(\delta) = f(\bigvee \Delta)$. Since Δ was arbitrary, f is continuous.

Proposition 1. *Let $(D, \leq)$ be a dcpo, and let id_D be the identity function on D. Then id_D is a continuous function.*

Proof. For $x, y \in D$ such that $x \leq y$, clearly $id_D(x) = x \leq y = id_D(y)$. Let $\Delta \subseteq_{dir} D$. We have $id_D(\bigvee \Delta) = \bigvee \Delta$, and $\bigvee id_D(\Delta) = \bigvee \Delta$. $\square$

Proposition 2. *Let $(D, \leq)$, $(D', \leq')$, and $(D'', \leq'')$ be dcpos, and let $f : D \to D'$ and $g : D' \to D''$ be continous functions. Then $g \circ f : D \to D''$ is continuous.*

Proof. Let $x, y \in D$ such that $x \leq y$. Then $f(x) \leq' f(y)$, and so $g(f(x)) \leq'' g(f(y))$. Thus, $g \circ f$ is monotonic. Let $\Delta \subseteq_{dir} D$. We need to show that $g(f(\bigvee \Delta)) = \bigvee g(f(\Delta))$. Since f is continuous, $g(f(\bigvee \Delta)) = g(\bigvee f(\Delta))$, and since g is continuous, $g(\bigvee f(\Delta)) = \bigvee g(f(\Delta))$. $\square$

1.2 The Categories Dcpo and Cpo

Propositions 1 and 2 allow us to define the following categories.

Definition 5. The category **Dcpo** has dcpos as objects and continuous functions as arrows. The category **Cpo** is the full subcategory of **Dcpo** with cpos as objects.

Checking in detail that **Dcpo** and **Cpo** are categories is left as an exercise. We now want to show that **Dcpo** and **Cpo** are cartesian closed categories (ccc), and thus induce λ-theories. We need to show the following:

1. There is a cpo $(T, \leq)$ such that for every dcpo $(D, \leq')$, there is exactly one continuous function $1_D : D \to T$. A cpo $(T, \leq)$ satisfying this condition is called a *terminal object*.
2. Given dcpos $(D, \leq)$ and $(D', \leq')$, there is a dcpo $(D \times D', \leq_\times)$ equipped with continuous functions π_1 and π_2 such that for any dcpo $(E, \leq''$ $)$ with continuous functions $f : E \to D$ and $g : E \to D'$, there

is a unique continuous function $\langle f, g\rangle : E \to D \times D'$, defined as $\langle f, g\rangle(x) = \langle f(x), g(x)\rangle$, such that $f = \pi_1 \circ \langle f, g\rangle$ and $g = \pi_2 \circ \langle f, g\rangle$. A dcpo $(D \times D', \leq_\times)$ equipped with continuous functions π_1 and π_2 is called a *product* of D and D'.

3. Given dcpos $(D, \leq)$, $(D', \leq')$, and $(E, \leq'')$, and a continuous function $f : D \times D' \to E$, there is a dcpo $(D' \to_{cont} E, \leq_{ext})$ equipped with continuous functions

$$curry(f) : D \to (D' \to E) \quad \text{and} \quad eval : ((D' \to E) \times D') \to E,$$

where $curry(f)$ is unique, such that $curry(f) \times id_{D'} : (D \times D') \to ((D' \to E) \times D')$ is continuous, and $f = eval \circ curry(f) \times id_{D'}$. A dcpo $(D' \to_{cont} E, \leq_{ext})$ equipped with continuous functions $curry(f)$ and $eval$ is called an *exponent* of D, D', and E.

It is easy to see that the cpo $(\{\bot\}, \leq_\bot)$ is a terminal object. The requirements 2. and 3. are more involved. We begin with 2.

Let $(D, \leq)$ and $(D', \leq')$ be dcpos. Define $(D \times D', \leq_\times)$ as follows: $D \times D'$ is the cartesian product of D and D', and $(x, x') \leq_\times (y, y')$ iff $x \leq y$ and $x' \leq' y'$. Moreover, define $\pi_1 : D \times D' \to D$ as $\pi_1(\langle x, x'\rangle) = x$, and $\pi_2 : D \times D' \to D'$ as $\pi_2(\langle x, x'\rangle) = x'$.

Proposition 3 (Products in **Dcpo**). *Let $(D, \leq)$ and $(D', \leq')$ be dcpos. Then $(D \times D', \leq_\times)$ with π_1 and π_2 is a product of D and D'.*

Proof. We first need to show that $(D \times D', \leq_\times)$ is a dcpo. Let $\Delta \subseteq_{dir} D \times D'$. Define $\Delta_D = \{x \mid \langle x, x'\rangle \in \Delta\}$ and $\Delta_{D'} = \{x' \mid \langle x, x'\rangle \in \Delta\}$. Then $\langle \bigvee \Delta_D, \bigvee \Delta_{D'}\rangle$ is the lub of Δ. To see why, let $\langle z, z'\rangle$ be an upper bound for Δ. Then, z is an upper bound for Δ_D and z' is an upper bound for $\Delta_{D'}$. Since $\bigvee \Delta_D$ is a lub for Δ_D, it follows that $\bigvee \Delta_D \leq z$. Similarly, $\bigvee \Delta_{D'} \leq' z'$. Thus, $\langle \bigvee \Delta_D, \bigvee \Delta_{D'}\rangle \leq_\times \langle z, z'\rangle$.

We also need to show that π_1 and π_2 are continuous. By symmetry, we only need consider π_1. Assume $\langle x, x'\rangle \leq_\times \langle y, y'\rangle$. Then $\pi_1(\langle x, x'\rangle) = x \leq y = \pi_1(\langle y, y'\rangle)$. Thus, π_1 is monotonic. Let $\Delta \subseteq_{dir} D \times D'$. Then $\pi_1(\bigvee \Delta) = \bigvee \Delta_D$. Moreover, since $\pi_1(\Delta) = \Delta_D$, it follows that $\bigvee \pi_1(\Delta) = \bigvee \Delta_D$. Thus π_1 is continuous.

Finally, let $(E, \leq'')$ be a dcpo with continuous functions $f : E \to D$ and $g : E \to D'$. We need to show that there is a unique function $\langle f, g\rangle : E \to$

$D \times D'$ such that $f = \pi_1 \circ \langle f, g \rangle$ and $g = \pi_2 \circ \langle f, g \rangle$. First, we show that $\langle f, g \rangle$ is continuous.

Let $x \leq'' y$. Since f and g are continuous, it follows that $f(x) \leq f(y)$ and $g(x) \leq' g(y)$. Thus, $\langle f(x), g(x) \rangle \leq_\times \langle f(y), g(y) \rangle$. Hence $\langle f, g \rangle$ is monotonic. Let $\Delta \subseteq_{dir} E$. Since $\langle f, g \rangle$ is monotonic, we have $\bigvee \langle f, g \rangle(\Delta) \leq_\times \langle f, g \rangle(\bigvee \Delta)$. Since f and g are continuous, we have the following:

$$\begin{aligned} \bigvee \{f(x) \mid x \in \Delta\} &= f(\bigvee \Delta) = f(\bigvee \Delta) \\ \bigvee \{g(x) \mid x \in \Delta\} &= g(\bigvee \Delta) = g(\bigvee \Delta). \end{aligned}$$

Let $\bigvee \langle f, g \rangle(\Delta) = \langle z, z' \rangle$. Then z is an upper bound for $\{f(x) \mid x \in \Delta\}$ and z' is an upper bound for $\{g(x) \mid x \in \Delta\}$. Thus $\bigvee \{f(x) \mid x \in \Delta\} \leq z$ and $\bigvee \{g(x) \mid x \in \Delta\} \leq' z'$. Hence $\langle f, g \rangle(\bigvee \Delta) \leq_\times \bigvee \langle f, g \rangle(\Delta)$. Extensionality ensures the uniqueness of $\langle f, g \rangle$. □

We have shown that the category **Dcpo** is closed under products. A simple addition to the argument shows that **Cpo** is also closed under products. We now move on to 3.

Let $(D, \leq)$ and $(D', \leq')$ be dcpos. Define $(D \to_{cont} D', \leq_{ext})$ as follows: $D \to_{cont} D'$ is the set of continuous functions from D to D', and $f \leq_{ext} g$ iff for all $x \in D$, $f(x) \leq' g(x)$.

We need the following lemmas. The proofs are left as an exercise.

Lemma 1. *Let $(D, \leq)$ and $(D', \leq')$ be dcpos. Then $(D \to_{cont} D', \leq_{ext})$ is a dcpo. If $(D, \leq)$ and $(D', \leq')$ are cpos, then $(D \to_{cont} D', \leq_{ext})$ is a cpo.*

Lemma 2. *Let $(D, \leq)$, $(D', \leq')$, and $(E, \leq'')$ be dcpos. A function $f : D \times D' \to E$ is continuous iff for all $x \in D$ ($y \in D'$) the functions $f_x : D' \to E$ ($f_y(x) : D \to E$) defined by $f_x(y) = f(\langle x, y \rangle)$ ($f_y(x) = f(\langle x, y \rangle)$) are continuous.*

Let $(D, \leq)$, $(D', \leq')$, $(E, \leq)$, and $(E', \leq')$ be dcpos, and let $f : D \to D'$ and $g : E \to E'$ be continuous functions. Define $f \times g : D(\times E) \to (D' \times E')$ as $f \times g(\langle x, x' \rangle) = \langle f(x), g(x') \rangle$.

Proposition 4. *The function $f \times g(\langle x, x' \rangle) = \langle f(x), g(x') \rangle$ is continuous.*

Proof. The proof is similar to that of $\langle f, g \rangle$. □

Proposition 5 (Exponents in **Dcpo**). *Let* $(D, \leq)$*,* $(D', \leq')$ *and* $(E, \leq'')$ *be dcpos, and let* $f : D \times D' \to E$ *be a continuous function. Then the dcpo* $(D' \to_{cont} E, \leq_{ext})$ *equipped with continuous functions*

$$curry(f) : D \to (D' \to E) \quad \text{and} \quad eval : ((D' \to E) \times D') \to E,$$

defined as $curry(f)(x)(y) = f(\langle x, y\rangle)$ *and* $eval(\langle f, x\rangle) = f(x)$*, respectively, is an exponent of* D*,* D'*, and* E*.*

Proof. We need to show that $curry(f)$ is continuous. Let $x \leq y$. Notice that $curry(f)(x) = f_x(x')$ and $curry(f)(y) = f_y(x')$ for $x' \in D'$. Let $x' \in D'$. Then $f_x(x') = f(\langle x, x'\rangle)$ and $f_y(x') = f(\langle y, x'\rangle)$. Since $\langle x, x'\rangle \leq_\times \langle y, x'\rangle$ and f is continuous, it follows that $f(\langle x, x'\rangle) \leq'' f(\langle y, x'\rangle)$. Since x' was arbitrary, $curry(f)(x) \leq_{ext} curry(f)(y)$.

To see that $curry(f)(\bigvee \Delta) \leq_{ext} \bigvee curry(f)(\Delta)$, let $x \in D'$. Then

$$curry(f)(\bigvee \Delta)(x') = f(\langle \bigvee \Delta, x'\rangle) = \bigvee f(\langle \Delta, x'\rangle) = \bigvee\{f(\langle \delta, x'\rangle) \mid \delta \in \Delta\}.$$

Let $g = \bigvee curry(f)(\Delta) = \bigvee\{curry(f)(\delta) \mid \delta \in \Delta\}$. Then for all $\delta \in \Delta$, $f(\langle \delta, x'\rangle) = curry(f)(\delta)(x') \leq'' g(x')$. Since x' was arbitrary, g is an upper bound for $\{f(\langle \delta, x'\rangle) \mid \delta \in \Delta\}$. Hence $\bigvee\{f(\langle \delta, x'\rangle) \mid \delta \in \Delta\} \leq_{ext} g$.

Since $curry(f)$ and $id_{D'}$ are continuous, it follows that $curry(f) \times id_{D'}$ is continuous.

We need to show that $eval$ is continuous. Let $\langle f, x\rangle \leq_\times \langle g, y\rangle$. Then, since $f \leq_{ext} g$, $eval(\langle f, x\rangle) = f(x) \leq'' g(x) \leq'' g(y) = eval(\langle g, y\rangle)$. Thus $eval$ is monotonic.

To show that $eval$ is continuous it is enough to show that $eval(\langle \_, x'\rangle)$ and $eval(\langle f, \_\rangle)$ are continuous for all $f \in D' \to E$ and $x' \in D$, respectively. By extensionality, $eval(\langle f, \_\rangle) = f(\_)$ for all $f \in (D' \to E)$, and is by assumption continuous.

Let $x' \in D'$. The monotonicity of $eval(\langle \_, x'\rangle)$ is trivial since $f \leq_{ext} g$ implies that $eval(\langle f, x'\rangle) = f(x') \leq'' g(x') = eval(\langle g, x'\rangle)$. Let $\Delta \subseteq_{dir} (D' \to E)$. We need to show that $eval(\langle \bigvee \Delta, x'\rangle) = \bigvee eval(\langle \Delta, x'\rangle)$. That is, $(\bigvee \Delta)(x') = \bigvee\{\delta(x') \mid \delta \in \Delta\}$. This follows from $f(\_) = \bigvee \Delta(\_)$ being the lub of Δ.

To see that $f = eval \circ curry(f) \times id_{D'}$, let $\langle x, x'\rangle \in D \times D'$. Then $curry(f) \times id_{D'}(\langle x, x'\rangle) = \langle curry(f)(x), x'\rangle$, and $eval(\langle curry(f)(x), x'\rangle) = f(\langle x, x'\rangle)$. $\square$

We have shown that **Dcpo** satisfies 1., 2. and 3. and so is a bona fide ccc. It is easy to augment the proofs above to directly show that **Cpo** is also a ccc. In the next section, we show that the abstract programming language PCF is naturally interpreted in **Cpo**.

2 Syntax and Semantics of PCF

3 Scott Topologies

3.1 Topologies and Bases

Before proceeding, we need a few definitions and concepts from basic topology.

Definition 6. Let X be a set. A *topology* on X is a set $T \subseteq \wp(X)$ such that $\emptyset$ and X are in T, and T is closed under infinite unions and finite intersections. A *topological space* is a set X together with a topology T on X, typically written as the ordered pair (X, T).

Let (X, T) be a topological space. The sets in T are called *open*, and the complements of open sets are called *closed*. For example, both X and $\emptyset$ are in T, and thus open. Moreover, $X^c = \emptyset$ and $\emptyset^c = X$, thus both X and $\emptyset$ are also closed. Sets that are both open and closed are called *clopen*.

Example 10. Let W be any set, and let $P = \wp(X)$. Then (W, P) is a topological space. Clearly, $\emptyset, W \in P$, by definition of powerset, and it is easy to verify that P is closed under both union and intersection.[2]

Definition 7. A *basis* over X is a collection $\mathfrak{B}$ of subsets of X such that

1. For each $x \in X$, there is at least one basis element B containing x.
2. If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

[2] Formal semanticists can think of W as the set of possible worlds and P as the set of all propositions.

Definition 8. If $\mathfrak{B}$ satisfies the two conditions in Definition 7, then we define the *topology generated by* $\mathfrak{B}$ as follows: A subset U of X is open if for each $x \in U$, there is a basis element $B \in \mathfrak{B}$ such that $x \in B$ and $B \subseteq U$. If T is the topology generated by $\mathfrak{B}$, then $\mathfrak{B}$ is called *a basis for* T.

Example 11. Let $Q = \{(q_1, q_2) \mid q_1, q_2 \in \mathbb{Q}, q_1 < q_2\}$. Then Q is a basis for $\mathbb{R}$. Indeed, let $r \in \mathbb{R}$. Then the open interval $(int(r) - 1, int(r) + 1) \in Q$ contains r[3]. Let $(q_1, q_2), (p_1, p_2) \in Q$ such that both contain r. If $(q_1, q_2) \subseteq (p_1, p_2)$, then (q_1, q_2) serves as the set we need. A similar argument holds if $(p_1, p_2) \subseteq (q_1, q_2)$, Assume that $(q_1, q_2) \nsubseteq (p_1, p_2)$, and without loss of generality, assume that $q_2 < p_2$. Then $r \in (p_1, q_2) \subseteq (q_1, q_2) \cap (p_1, p_2)$.

Proposition 6. *Let X be a set, and let $\mathfrak{B}$ be a basis for a topology T on X. Then T is the collection of all unions of elements of $\mathfrak{B}$.*

Example 12. Given Example 11, we have that $\mathfrak{Q} = \{\bigcup Q' \mid Q' \subseteq Q\}$ is the topology generated by Q. Moreover, $(\mathbb{R}, \mathfrak{Q})$ is a topological space.

Definition 9. Let (X, T) and (Y, S) be topological spaces. A function $f : X \to Y$ is called *continuous* if, for all open sets $U \in S$, $f^{-1}(U) \in T$.

Example 13. Let $(\mathbb{R}, \mathfrak{Q})$ be the topological space in Example 12, and let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$. Then f is continuous (in the topological sense). To see why this is, consider an open set $U = (4, 25)$. Then $f^{-1}(U) = (-5, -2) \cup (2, 5)$. Since $(-5, -2)$ and $(2, 5)$ are open, so is $f^{-1}(U)$. Thus the inverse image of U is open. This is true in general, thus f is continuous.

3.2 Scott Topologies

We now look at topologies based on partial orders. Let $(D, \leq)$ be a partial order. A subset $A \subseteq D$ is *upper closed under* $\leq$. if, for $x \in A$ and $x \leq y$, it follows that $y \in A$. The *Alexandrov topology on D*, denoted $\mathfrak{A}$, is the set of upper closed subsets of D. That is, $(D, \mathfrak{A})$ is a topological space.

We can also recover partial orders from (T_0) topologies. Let (X, T) be a (T_0) topological space. Define the *specialization order on X* as $x \leq y$ iff for all $U \in T$, if $x \in U$, then $y \in U$. It is easy to verify that $\leq$ is a partial order (assuming T is T_0).

[3] $int(r)$ is the integer portion of r.

We now define topologies that are based on dcpos.

Definition 10. Let $(D, \leq)$ be a dcpo. A subset $A \subseteq D$ is called *Scott open* if the following hold:

- A is *upper closed under* $\leq$, i.e. if $x \in A$ and $x \leq y$, then $y \in A$.
- If $\Delta \subseteq$ is directed and $\bigvee \Delta \in A$, then there is an $x \in \Delta$ such that $x \in A$.

Proposition 7. *Let $(D, \leq)$ be a dcpo and let Ω_D be the set of Scott open subsets of D. Then (D, Ω_D) is a topological space.*

Proof. Clearly, $\emptyset, D \in \Omega_D$. Assume $S \subseteq \Omega_D$ and let $x \in \bigcup S$. Then $x \in A$ for some $A \in S$. Let $y \in D$ such that $x \leq y$. Since A is upper closed under $\leq$, it follows that $y \in A$. Thus $y \in \bigcup S$. Let $\Delta \subseteq_{dir} D$ such that $\bigvee \Delta \in \bigcup S$. Then $\bigvee \Delta \in A$ for some $A \in S$. Since A is Scott open, there is a $z \in \Delta$ such that $z \in A$. Thus $z \in \bigcup S$.

Assume S is of finite cardinality. Let $x \in \bigcap S$. Then $x \in A$ for each $A \in S$. Let $y \in D$ such that $x \leq y$. Since each A is upper closed under $\leq$, it follows that $y \in A$, for each A. Thus $y \in \bigcap S$. Let $\Delta \subseteq_{dir} D$ such that $\bigvee \Delta \in \bigcap S$. Then $\bigvee \Delta \in A$ for each $A \in S$. Since A is Scott open, for each A, there is a $z_A \in \Delta$ such that $z_A \in A$. Since Δ is directed, there is a $z \in \Delta$ such that $z_A \leq z$ for each z_A. Thus $z \in A$ for each A. Hence $z \in \bigcap S$. □

The topological space (D, Ω_D) is called the *Scott topology* over D.

Lemma 3. *Let $(D, \leq)$ be a dcpo. The specialization order $\leq'$ on (D, Ω_D) is the partial order $\leq$.*

Proof. We need to show set equality of the relations $\leq'$ and $\leq$. Let $(x, y) \in \leq$, and let $U \in \Omega_D$. Suppose $x \in U$. Since U is upper closed under $\leq$, and $x \leq y$, it follows that $y \in U$. That is, $(x, y) \in \leq'$. Thus, $\leq \subseteq \leq'$. Now, let $(x', y') \in \leq'$. Notice that $U_{y'} = \{a \in D \mid a \not\leq y'\}$ is Scott open, and hence in Ω_D. Suppose $x' \not\leq y'$. Then $x' \in U_{y'}$, and since $(x', y') \in \leq'$, it follows that $y' \in U_{y'}$. Yet, this is a contradiction. Thus $x' \leq y'$, and so $\leq' \subseteq \leq$. □

Lemma 4. *Let $(D, \leq)$ and $(D', \leq')$ be dcpos. The (topologically) continuous functions from (D, Ω_D) to $(D', \Omega_{D'})$ are the (order theoretic) continuous functions from $(D, \leq)$ to $(D', \leq')$.*

Proof. Let $f : D \to D'$ be (topologically) continuous. We need to show that f is monotonic. Assume $x \leq y$. We need to show that $f(x) \leq' f(y)$. Let $U' \in \Omega_{D'}$ be any open set containing $f(x)$. Now, $f^{-1}(U')$ is an open set containing x, and thus contains y. Hence, $f(y) \in U'$. Since U' was arbitrary, it follows that $x \leq' y$.

We need to show that $f(\bigvee \Delta) = \bigvee f(\Delta)$, for directed Δ. By monotonicity, we immediately have that $\bigvee f(\Delta) \leq' f(\bigvee \Delta)$. Suppose $f(\bigvee \Delta) \not\leq' \bigvee f(\Delta)$. Then $f(\bigvee \Delta) \in U_{\bigvee f(\Delta)} = \{a \in D' \mid a \not\leq' \bigvee f(\Delta)\}$. Thus, $\bigvee \Delta \in f^{-1}(U_{\bigvee f(\Delta)})$, and so $f(\delta) \in U_{\bigvee f(\Delta)}$ for some $\delta \in \Delta$. This is a contradiction since $f(\delta) \leq' \bigvee f(\Delta)$. □

3.3 Algebraicity and Partial Continuity

Definition 11. Let $(D, \leq)$ be a dcpo. an element $d \in D$ is called *compact* if, for each $\Delta \subseteq_{dir} D$, from $d \leq \bigvee Delta$ it follows that there is $x \in \Delta$ such that $d \leq x$.

We denote the set of compact elements of D by $K(D)$. That is, $K(D) = \{d \in D \mid d$ is compact $\}$. The set $K(D)$ is called the *basis* of D. We will show that the elements of $K(D)$ indeed yield a basis for the Scott topology on (D, Ω_D), when the following definition holds on $(D, \leq)$.

Definition 12. Let $(D, \leq)$ be a dcpo. Then $(D, \leq)$ is called *algebraic* if for all $x \in D$ the set $K(D)_x = \{d \in K(D) \mid d \leq x\}$ is directed, and $\bigvee K(D)_x = x$.

The elements of $K(D)_x$ are called *approximants* of x. To see why, consider the following example.

Example 14. The dcpo $(\wp(\omega), \subseteq)$ is algebraic. Let Δ_{odd} be the set of all sets of odd numbers. Then $\bigvee \Delta_{odd} = \{a \in \omega \mid a$ is odd $\}$. Now, $\bigvee \Delta_{odd}$ is an infinite set, and $K(D)_{\bigvee \Delta_{odd}}$ is the set of all finite sets of odd numbers. That is, $K(D)_{\bigvee \Delta_{odd}} = \{\{1\}, \{1, 3\}, \{1, 3, 5\}, \ldots\}$. Each set in $K(D)_{\bigvee \Delta_{odd}}$ is an approximation of $\bigvee \Delta_{odd}$, and as the sets in $K(D)_{\bigvee \Delta_{odd}}$ get larger, they better approximate $\bigvee \Delta_{odd}$.

Proposition 8. *Let $(D, \leq)$ be and algebraic dcpo. Let $\uparrow d = \{x \in D \mid d \leq x\}$. Then $B = \{\uparrow d \mid d \in K(D)\}$ is a basis for (D, Ω_D).*

Proof. We need to show that B is a basis. Let $x \in D$. Since D is algebraic, $x = \bigvee K(D)_x$, and so $x \in \uparrow d$ for each $d \in K(D)_x$. Let $x \in \uparrow d \cap \uparrow e$. Then $d, e \in K(D)_x$. Since $K(D)_x$ is directed, there is a $z \in K(D)_x$ such that $d \leq z$ and $e \leq z$. Thus $\uparrow z \subseteq \uparrow d$ and $\uparrow z \subseteq \uparrow e$. Thus $\uparrow z \subseteq (\uparrow d \cap \uparrow e)$.

Finally, notice that each $\uparrow d$ is Scott open. Indeed, let $\bigvee \Delta \in \uparrow d$. Then $d \leq \bigvee \Delta$, and by compactness of d, there is a $z \in \Delta$ such that $d \leq z$. Hence $z \in \uparrow d$. Let $A \in \Omega_D$, and let $x \in A$. Since D is algebraic, and since $\bigvee K(D)_x = x \in A$, there is a compact d such that $d \in A$. Thus, $\uparrow d \subseteq A$. Thus Ω_D is the topology generated by B. □

We conclude this section with a definition of partial continuous function, and its role in the categorical theory of dcpos. Let X and Y be sets. A *partial function* $f : X \to Y$ is a function $f : X' \to Y$ such that $X' \subseteq X$. The set X' is called the *domain* of f, and is denoted $dom(f)$.

Definition 13. Let $(D, \leq)$ and $(D', \leq')$ be dcpos. A partial function $f : D \to D'$ is called *continuous* if $dom(f)$ is Scott open, and f restricted to $dom(f)$ is continuous. That is, $f : dom(f) \to D'$ is continuous.

We will use the following category in our discussion of monads. For now, we simply give a definition.

Definition 14. The category **pDcpo** has dcpos as objects and partial continuous functions as arrows.

4 The syntax and semantics of control

A The λ-Calculus

Much of the following material is drawn from Gunter (1992). Let Σ_1 be a collection of type constants, or *basic types*. We form the *types over* Σ_1 with the context-free grammar:

$$\mathcal{S} ::= 1 \mid A \mid \mathcal{S} \to \mathcal{S} \mid \mathcal{S} \times \mathcal{S}$$

where $A \in \Sigma_1$, and 1 is the null product type. That is, the types over Σ_1 are the trees generated by this grammar.

Let Σ_0 be a function from term constants to types over Σ_1. The pair $\Sigma = (\Sigma_0, \Sigma_1)$ is called a *signature*. We form the *terms over* Σ_0 with the context-free grammar:

$$\mathcal{T} ::= * \mid a \mid x \mid \lambda_x \mathcal{T} \mid \mathcal{T}\mathcal{T} \mid (\mathcal{T}, \mathcal{T}) \mid \pi(\mathcal{T}) \mid \pi'(\mathcal{T})$$

where x is a variable and $(a, A) \in \Sigma_0$. The $*$ is a special constant of type 1. The terms over Σ_0 are the trees generated by this grammar, called *term trees*. Equivalence classes of term trees modulo $\equiv_\alpha$ are called *λ-terms*, or simply *terms*. Free variables and substitution are defined in the usual way.

Herein we use Latin minuscules as metavariables ranging over λ-terms. Let $\Sigma = (\Sigma_0, \Sigma_1)$ be a signature. A *type assignment* is a (possibly empty) list Γ of pairs $x : A$, where x is a variable and A a type, such that the variables in Γ are distinct.

A *typing judgment* is a triple consisting of a type assignment Γ, a term a, and a type A such that all of the free variables of a occur in Γ. Let $\mathcal{A}$ be the collection of all type assignments, and let Λ_0 be the collection of all λ-terms and Λ_1 be the collection of all types. Let

$$\mathcal{J} = \{(\Gamma, a, A) \in \mathcal{A} \times \Lambda_0 \times \Lambda_1 \mid \text{ all the free variables in } a \text{ occur in } \Gamma\}.$$

That is, $\mathcal{J}$ is the collection of all typing judgments. We define $\rhd : \subseteq \mathcal{J}$ to be the least relation closed under the axioms and rules in Table 1. We write $\Gamma \rhd a : A$ to indicate that $(\Gamma, a, A) \in \rhd :$. If Γ is empty, we write $\rhd a : A$. The typing judgments that appear above the line in each rule are called *premises*, and those below the line are called *conclusions*.

A *typing derivation* is a labelled tree where the labels are typing judgements, the leaves are axioms and each non-leaf is labelled by the conclusion of a rule whose premises are the labels of that non-leaf's daughters. A term a is of type A iff $\Gamma \rhd a : A$ is the conclusion of some typing derivation. If $\Gamma \rhd a : A$ and $\Gamma \rhd a : A'$, then $A = A'$.

An *equation* is a four-tuple (Γ, a, b, A) where Γ is a type assignment, $\Gamma \rhd a : A$ and $\Gamma \rhd b : A$. We typically write equations as $(\Gamma \rhd a = b : A)$. Let T be a set of equations. We write $T \vdash (\Gamma \rhd a = b : A)$ if $(\Gamma \rhd a = b : A) \in T$. The

V-PROJECTION	$\Gamma, x : A, \Gamma' \rhd x : A$	PAIRING	$\dfrac{\Gamma \rhd a : A \qquad \Gamma \rhd b : B}{\Gamma \rhd (a, b) : A \times B}$
CONSTANT	$\Gamma \rhd c : \Sigma_0(c)$	FIRST	$\dfrac{\Gamma \rhd h : A \times B}{\Gamma \rhd \pi(h) : A}$
NULL PRODUCT	$\Gamma \rhd * : 1$	SECOND	$\dfrac{\Gamma \rhd h : A \times B}{\Gamma \rhd \pi'(h) : B}$
ABSTRACTION	$\dfrac{\Gamma, x : A \rhd b : B}{\Gamma \rhd \lambda_x b : A \to B}$	PERMUTE	$\dfrac{\Gamma, x : B, y : C, \Gamma' \rhd a : A}{\Gamma, y : C, x : B, \Gamma' \rhd a : A}$
APPLICATION	$\dfrac{\Gamma \rhd f : A \to B \qquad \Gamma \rhd a : A}{\Gamma \rhd f(a) : B}$	WEAKENING	$\dfrac{\Gamma \rhd a : A \quad (x \notin \Gamma)}{\Gamma, x : B \rhd a : A}$

Table 1: Typing rules

statement $T \vdash (\Gamma \rhd a = b : A)$ is called an *equational judgment*. We write $\vdash (\Gamma \rhd a = b : A)$ to indicate that $(\Gamma \rhd a = b : A)$. is to be included in every theory.

An *equational theory* is a set of equations closed under the axioms and rules in Table 2. The equational judgments that appear above the line in each rule are called *premises*, and those below the line are called *conclusions*. An *equational derivation* is defined exactly as a typing derivation, with equational judgements in place of typing judgements.

Definition 15. An equational theory that satisfies the rules in Table 3 is called a *λ-theory*.

References

Amadio, R. M., & Curien, P. L. (1998). *Domains and Lambda-Calculi*. Cambridge University Press.

Gunter, C. (1992). *Semantics of Programming Languages*. The MIT Press.

Add	$\dfrac{T \vdash (\Gamma \rhd a = b : A) \quad (x : B \notin \Gamma)}{T \vdash (\Gamma, x : B \rhd a = b : A)}$
Drop	$\dfrac{T \vdash (\Gamma, x : B \rhd a = b : A) \quad (x \notin \mathrm{Fv}(a) \cup \mathrm{Fv}(b))}{T \vdash (\Gamma \rhd a = b : A)}$
Permute	$\dfrac{T \vdash (\Gamma, x : B, y : C, \Gamma' \rhd a = b : A)}{T \vdash (\Gamma, y : C, x : B, \Gamma' \rhd a = b : A)}$
Reflexivity	$\vdash (\Gamma \rhd a = a : A)$
Symmetry	$\dfrac{T \vdash (\Gamma \rhd a = b : A)}{T \vdash (\Gamma \rhd b = a : A)}$
Transitivity	$\dfrac{T \vdash (\Gamma \rhd a = b : A) \qquad T \vdash (\Gamma \rhd b = c : A)}{T \vdash (\Gamma \rhd a = c : A)}$
μ	$\dfrac{T \vdash (\Gamma \rhd a = b : A \to B) \qquad T \vdash (\Gamma \rhd c = d : A)}{T \vdash (\Gamma \rhd a(c) = b(d) : B)}$
ξ	$\dfrac{T \vdash (\Gamma, x : A \rhd a = b : B)}{T \vdash (\Gamma \rhd \lambda_x a = \lambda_x b : A \to B)}$

Table 2: Equational Rules

Null Product	$\vdash (\Gamma \rhd a = * : 1)$
β	$\vdash (\Gamma \rhd (\lambda_x b)(a) = [a/x]b : B)$
η	$\vdash (\Gamma \rhd \lambda_x f(x) = f : A \to B) \quad (x \notin \mathrm{Fv}(f))$
F-projection	$\vdash (\Gamma \rhd \pi(a, b) = a : A)$
S-projection	$\vdash (\Gamma \rhd \pi'(a, b) = b : B)$
Pair-identity	$\vdash (\Gamma \rhd (\pi(a, b), \pi'(a, b)) = (a, b) : A \times B)$

Table 3: λ-rules