In Chapter 2 we have seen several operations on convex combinations (cf. Example 2.11.1).

Definition 2.1. In a ge-convex set $A$, the intersection of $A$ with a polytope $F$ is denoted $A \cap F$.

Theorem 2.2. If $A$ is a ge-convex set and $F$ is a polytope, then $A \cap F$ is a ge-convex set.

Proof. The intersection of a ge-convex set with a polytope is a ge-convex set.

Theorem 2.3. If $A$ is a ge-convex set and $F$ is a polytope, then $A \cap F$ is a ge-convex set.

Proof. The intersection of a ge-convex set with a polytope is a ge-convex set.

Theorem 2.4. If $A$ is a ge-convex set and $F$ is a polytope, then $A \cap F$ is a ge-convex set.

Proof. The intersection of a ge-convex set with a polytope is a ge-convex set.

Theorem 2.5. If $A$ is a ge-convex set and $F$ is a polytope, then $A \cap F$ is a ge-convex set.

Proof. The intersection of a ge-convex set with a polytope is a ge-convex set.
6.2 Fixpoint Induction

The Lague PFP

Exercise 6.2.1 Consider the extension of the simply typed lambda-calculus with a natural number object. Is this

\[ f = \frac{\lambda x. (\lambda y. x y) (\lambda y. y) \varepsilon}{\lambda x. (\lambda y. x y) (\lambda y. y) \varepsilon} \]

\[ \varepsilon = \frac{\varepsilon}{\lambda x. \varepsilon} \]

Hint: Show that this function is total.
6.3. The Programming Language PDP

Definition 6.3.1 (PDP)

We define PDP as a standard model, which has become a widely popular and influential approach to designing programming languages. PDP is an extension of DPC, which has been developed in recent years. In this section, we will focus on the introduction of the language PDP, which is an extension of DPC.

Figure 6.1: The constants of PDP

θ : θ α : α

Figure 6.2: The variables of PDP

θ : θ α : α

Figure 6.3: The operations of PDP

θ : θ α : α

Definition 6.3.2 (PDP)

The model in which the language PDP is defined is called the PDP model. In this model, the language PDP is defined as follows:

\[ (\text{PDP}) = \{ \text{PDP} \} \]

Definition 6.3.3 (PDP)

The PDP model is defined as follows:

\[ (\text{PDP}) = \{ \text{PDP} \} \]

In this model, the language PDP is defined as follows:

\[ (\text{PDP}) = \{ \text{PDP} \} \]
Theorem 6.3. The operational semantics for PDP is associative, commutative, and distributes over parallel composition.

Proof. By the definition of the operational model, we have

\[ (\text{some expression}) \Rightarrow \text{some other expression} \]

Since the expression is a defining the semantics of PDP, we can conclude that it holds for all possible inputs.

Example: Consider the expression \( (x + y) \cdot (z + w) \).

By definition of parallel composition, we have

\[ (x + y) \cdot (z + w) \Rightarrow x + (y \cdot (z + w)) \]

This shows that the operational semantics is associative.

By definition of parallel composition, we have

\[ (x \cdot y) \cdot (z \cdot w) \Rightarrow x \cdot (y \cdot (z \cdot w)) \]

This shows that the operational semantics is commutative.

Example: Consider the expression \( (x + y) + (z + w) \).

By definition of parallel composition, we have

\[ (x + y) + (z + w) \Rightarrow (x + (y + z)) + w \]

This shows that the operational semantics distributes over parallel composition.
6.4 The full abstraction problem for PCD

The language PCD

Definition 6.4.1: (full abstraction) A PCD model of a language L is a full abstraction if, for any PCD program P:

P = [P]L

Definition 6.4.2: (full abstraction) A PCD model of a language L is a full abstraction if, for any PCD program P:

P = [P]L

The result is a full abstraction of the PCD model of (PCD version of) the language L.

This result is an extension of the previous work of (PCD version of) the language L.

Then there exists an extension of the previous work of (PCD version of) the language L.

We define a preorder 0 on a PCD model M of a language L as follows:

Definition 6.4.3: (preorder on a PCD model) A PCD model M of a language L is a full abstraction if, for any PCD program P:

P = [P]L

We prove that the PCD model M of a language L is a full abstraction if, for any PCD program P:

P = [P]L

In section 2.2 we discuss the full abstraction result for PCD. Then, all other cases are covered in the next section.

For more details, please refer to [1].
Theorem 6.2. Towards separability

In Section 5, we have shown that the problem of finding a separable solution for the set of constraints is equivalent to finding a solution to the following problem:

\[ \min_{x \in \mathbb{R}^n} \{ f(x) \mid A x \leq b, \ \text{subject to} \ \Omega \} \]

where \( f(x) \) is the objective function, \( A \) is the constraint matrix, and \( \Omega \) is the feasible region.

We have already seen that the Karush-Kuhn-Tucker conditions are necessary and sufficient for a solution to be optimal.

Remark 6.2.1. For the separable case, the problem can be solved efficiently using primal-dual interior-point methods.

Example 6.2.2. Consider the following problem:

\[ \min_{x \in \mathbb{R}^2} \{ x_1^2 + x_2^2 \} \quad \text{subject to} \quad x_1 + x_2 = 1 \]

The optimal solution is \( x^* = (0.5, 0.5) \).

Theorem 6.2.1. (Separation result) Suppose that the feasible region \( \Omega \) is convex and compact. Then there exists a separating hyperplane for \( \Omega \) if and only if \( \Omega \) is not contained in any hyperplane.

Proof. Suppose that \( \Omega \) is not contained in any hyperplane. Then there exists a separating hyperplane for \( \Omega \).

The converse is also true. If there exists a separating hyperplane for \( \Omega \), then \( \Omega \) cannot be contained in any hyperplane.

Example 6.2.3. Consider the following problem:

\[ \min_{x \in \mathbb{R}^2} \{ x_1^2 + x_2^2 \} \quad \text{subject to} \quad x_1 + x_2 = 1 \]

The optimal solution is \( x^* = (0.5, 0.5) \).

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The optimal solution is \( x^* = (0.5, 0.5) \).

Theorem 6.2.4. (Separation result) Suppose that the feasible region \( \Omega \) is convex and compact. Then there exists a separating hyperplane for \( \Omega \) if and only if \( \Omega \) is not contained in any hyperplane.

Proof. Suppose that \( \Omega \) is not contained in any hyperplane. Then there exists a separating hyperplane for \( \Omega \).

The converse is also true. If there exists a separating hyperplane for \( \Omega \), then \( \Omega \) cannot be contained in any hyperplane.